Different measures of graph expansion

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Rumor spreading: Analysis for general graphs

1 Different measures of graph expansion

We formally state several notions of graph expansion.

• Spectral: For a regular graph, $1 - \lambda_2$, where $\lambda_2$ is the second largest eigenvalue of the random walk matrix. Alternatively, the second smallest eigenvalue of the Laplacian.

• Vertex expansion: Let $N(S)$ denote the set of nodes in $V - S$ that have at least one neighbor in $S$. The vertex expansion is defined as:

\[
\min_{S : 0 < |S| \leq |V|/2} \frac{|N(S)|}{|S|}.
\]

• Edge expansion: Let $E(S, V - S)$ denote the set of edges with one endpoint in $S$ and the other endpoint in $V - S$. The edge expansion is defined as:

\[
\min_{S : 0 < |S| \leq |V|/2} \frac{|E(S, V - S)|}{|S|}.
\]

• Conductance: Let $\text{vol}(S)$ denote the sum of the degrees of all the vertices in $S$. The conductance $\Phi$ is defined as:

\[
\min_{\emptyset \neq S \neq V} \frac{|E(S, V - S)|}{\min\{\text{vol}(S), \text{vol}(V - S)\}}.
\]

A well-known result – Cheeger’s Inequality – relates conductance with the spectral graph.

**Theorem 1.** Cheeger For any $d$-regular graph, we have

\[
\frac{\Phi^2}{8} \leq 1 - \lambda_2 \leq \Phi.
\]

2 Random bipartite graphs expand

Consider a random bipartite graph $G = (L \cup R, E)$, where $L$ and $R$ are two disjoint sets of $n$ vertices, and $E$ is a collection of edges between $L$ and $R$ drawn as follows: each vertex in $L$ selects $d$ vertices in $R$ uniformly at random, and independent of choices of any other vertex in $L$. Clearly, every vertex in $L$ has exactly $d$ incident edges. We ask the following question: how well does an arbitrary set in $L$ expand?
**Theorem 2.** With probability at least $1/2$, every subset $S$ of vertices in $L$ with $|S| \leq n/d$ has at least $d|S|/4$ neighbors in $R$.

**Proof:** Consider any set $S$ of size $s \leq n/d$ vertices. The probability that it has fewer than $ds/4$ neighbors is at most

$$\binom{n}{ds/4} \left(\frac{ds/4}{n/d}\right)^s.$$

Simplifying, using the approximation $(ds/d)/(n/d) \approx (ds/4n)^d$, and using the inequality $(a/b)^d \leq (ea/b)^b$ for $a \geq b > 0$, we obtain the bound:

$$\left(\frac{4en}{ds}\right)^{3ds/4} \cdot \left(\frac{ds}{4n}\right)^{ds} \leq \left(\frac{ed}{4n}\right)^{3ds/4}.$$

Since $ds \leq n$, we obtain that the probability that there exists any set $S$ of size at most $n/d$ that has fewer than $d|S|/4$ neighbors is at most

$$\sum_{s \geq 1} \frac{e^{3ds/4}}{4} \leq 1/2$$

for $d \geq 4$. \qed

**3 Analysis of rumor spreading on general graphs**

The paradigm of rumor spreading or gossiping is considered as a robust mechanism for spreading information in a distributed network, or influence in a social network. Suppose we have an undirected connected network $G$ with $n$ nodes. A node, say $r$, has a piece of information $M$ that it wants to broadcast to the entire network. Consider the following gossiping protocol.

In each step, each node that has a copy of $M$, sends a copy of $M$ to a neighbor chosen uniformly at random. Assume that all the nodes are synchronized in their steps. This is called the Push protocol.

**Theorem 3.** The Push protocol completes in $O(n \log n)$ steps with probability at least $1 - 1/n$ for any $n$-vertex graph.

**Proof:** Our proof follows the following steps.

(a) Suppose a node $u$ has a copy of $M$ and degree $d$. What is the expected number of steps, in terms of $d$, before $u$ sends a copy of $M$ to a specific neighbor $v$?

The probability that $u$ sends a copy of $M$ to $v$ in any given step is $1/d$. Thus, the expected number of steps it takes before $u$ sends a copy of $M$ to $v$ equals:

$$\frac{1}{d} + 2 \cdot \frac{1}{d} \cdot \frac{d-1}{d} + 3 \cdot \frac{1}{d} \cdot \left(\frac{d-1}{d}\right)^2 + \ldots + \sum_{i=1}^{\infty} i \cdot \frac{1}{d} \left(\frac{d-1}{d}\right)^{i-1}.$$

Using elementary algebra/calculus, we simplify the above to obtain the expectation to be $d$. 2
(b) Let $P$ be a shortest path from $u$ to $v$. We now show that the sum of the degrees of all the nodes on $P$ is at most $3n$. We argue that a node $x$ can be a neighbor of at most 3 nodes on a shortest path. Note that this is sufficient to establish the desired claim.

Suppose otherwise; let $x$ be a neighbor of distinct nodes $u_1$, $u_2$, $u_3$, and $u_4$. Without loss of generality, assume that $P$ first visits $u_1$, then $u_2$, then $u_3$, and then $u_4$. It follows that the subpath of $P$ from $u_1$ to $u_4$ has at least three edges. However, replacing this subpath by the two-hop path $u_1 \rightarrow x \rightarrow u_4$ contradicts the fact that $P$ is a shortest path from $u$ to $v$.

(c) Using parts (a) and (b), we now derive an upper bound, in terms of $n$, on the expected number of steps it takes for an arbitrary node $v$ to receive a copy of $M$.

By part (a) and linearity of expectation, the expected number of steps it takes for an arbitrary node $v$ to receive a copy of $M$ is at most the sum of the degrees of the nodes along the shortest path from $r$ to $v$, which is at most $3n$ by part (b).

Unfortunately, part (c) does not give us a bound on the expected completion time, since it only bounds the time taken for an arbitrary node $v$ – not all nodes – to receive $M$.

(d) Let us revisit part (b). Again, suppose a node $u$ has a copy of $M$ and degree $d$. We find an upper bound, in terms of $d$, on the number of steps it takes for a specific neighbor $v$ of $u$ to receive a copy of $M$ with probability at least $1 - 1/n^3$.

Let $t$ be the number of steps it takes for $v$ to receive a copy of $M$ from $u$ with probability at least $1 - 1/n^3$. The probability that $v$ has not received a copy of $M$ from $u$ in $t$ steps is $(1 - 1/d)^t$. So $t$ is the first step at which this probability is at most $1/n^3$; in other words

$$t \leq \ln(1/n^3)/\ln(1 - 1/d) \leq 3d\ln n,$$

where we use the inequality $(1 - 1/d)^d \leq 1/e$ for $d \geq 1$.

(e) Using parts (b) and (d), we derive an upper bound, in terms of $n$, on the number of steps it takes for an arbitrary node $v$ to receive a copy of $M$ with probability at least $1 - 1/n^2$. We argue that the same bound yields an upper bound on the number of steps it takes for all nodes to receive a copy of $M$ with probability at least $1 - 1/n$.

Consider a shortest path from $r$ to $v$. In at most $3d_r \ln n$ steps, where $d_r$ is the degree of $r$, the message crosses the first hop (to, say node $u$) with probability at least $1 - 1/n^3$. Conditioned on the fact that $M$ has reached $u$, in at most $3d_u \ln n$ additional steps, where $d_u$ is the degree of $u$, the message crosses the second hop with probability at least $1 - 1/n^3$. Thus, in at most $3(d_r + d_u) \ln n$ steps, $M$ has reached $u$ with probability at least $1 - 2/n^3$ (using Boole’s inequality). Continuing with this argument and invoking part (b), we obtain that $M$ reaches an arbitrary node $v$ in at most $3n \log n$ steps with probability at least $1 - n/n^3 = 1 - 1/n^2$. The probability that $M$ has failed to reach a specific node $v$ in $3n \log n$ steps is at most $1/n^2$. Thus, the probability that there exists a node $v$ that $M$ has failed to reach in $3n \log n$ steps is at most $1/n$ (using Boole’s inequality).