- Random walks and electrical networks
- Random walk measures and graph parameters

1 Random walks and electrical networks

We present an intriguing connection between random walks and electricity flow in networks. This phenomena has been known for several years, going back to the nineteenth century. A comprehensive manuscript on this connection is by Doyle and Snell, available online. Our coverage of this connection will closely follow Lovasz's survey on random walks. We will introduce the notion of effective resistance between two nodes in a graph, which has found several applications recently, including in graph sparsification. More recently, the connection between electrical flows and random walks has been instrumental in obtaining near-linear time algorithms for network flows — considered among the breakthrough results in the field.

Consider an undirected graph G, and let s and t denote distinct vertices in G. Now, view this graph G as an electrical network: assume that each edge corresponds to a resistor of resistance 1Ω, and there is current i entering s and exiting t such that the difference in voltage between t and s is 1. Specifically, we let the voltage $\phi(s)$ at s be 0, and the voltage at $\phi(t)$ at t be 1. Let $\phi(v)$ denote the voltage at any vertex v. Let i_{xy} denote the current flowing from x to y; note that $i_{yx} = -i_{xy}$.

By Ohm's Law, we have, for every edge (u, v):

$$\phi(v) - \phi(u) = i_{uv}.$$

Adding the above equation, over all incident edges of a vertex $v \neq s, t$, we obtain

$$d(v)\phi(v) - \sum_{(u,v)\in E} \phi(u) = \sum_{(u,v)\in E} i_{uv} = 0,$$

where the last equality follows from Kirchoff's Law. We thus have

$$\phi(v) = \frac{1}{d(v)} \sum_{(u,v)\in E} \phi(u) \tag{1}$$

A function f satisfying Equation 1 and the conditions $\phi(s) = 0, \phi(t) = 1$ is referred to as a harmonic function with poles s and t.

We now establish that an appropriately defined probability function for a random walk is also a harmonic function with poles s and t. Let $p(u)$ denote the probability that a random walk starting at u visits t before s. Clearly, $p(s) = 0$ and $p(t) = 1$. For any $v \neq s, t$, we obtain

$$p(v) = \frac{1}{d(v)} \sum_{(u,v)\in E} p(u).$$

We leave the following observation as an exercise.

Observation 1. *There is a unique function satisfying Equation 1.*

We thus obtain that ϕ and p are the same functions. This equivalence leads to the following beautiful result connecting the commute times of a random walk and effective resistances. For any two vertices s and t, the *commute time* $C(s,t)$ is the sum of the hitting times $H(s,t) + H(t,s)$. Note that unlike hitting time, commute time is symmetric, and, in fact, forms a metric. We define the *effective resistance* R_{st} between s and t in G as the potential difference between s and t if a unit current enters the network at s and leaves at t; alternatively, it is the inverse of the current i that enters s and leaves t if the potential difference between s and t is set to 1. That is:

$$R_{st} = \frac{1}{\sum_{(s,v)\in G} i_{sv}} = \frac{1}{\sum_{(s,v)\in G} \phi(v)} = \frac{1}{\sum_{(s,v)\in G} p(v)} \tag{2}$$

Theorem 1. *For any graph G and vertices s and t, we have*

$$C(s,t) = 2mR_{st}.$$

Proof: Let p denote the probability that a random walk starting at s visits t before returning to t. Let τ denote the return time to s, and let σ denote the time it takes to return to u after visiting t. Clearly $\tau \le \sigma$. We know that

$$E[\tau] = \frac{2m}{d(s)}$$

and

$$E[\sigma] = C(s,t).$$

We also have the following relationship between σ and τ.

$$E[\sigma - \tau] = (1-p)E[\sigma].$$

Simplifying yields

$$p = \frac{E[\tau]}{E[\sigma]} = \frac{2m}{d(s)C(s,t)}.$$

Now p is also given as follows.

$$p = \frac{1}{d(s)} \sum_{(s,v)\in E} p_v = \frac{1}{R_{st}d(s)},$$

where the last equality follows from Equation 2. Putting the preceding two equations for p together yields $C(s,t) = 2mR_{st}$. □

2 Random walks and graph measures

We have seen that the time taken for a random walk to mix within ε distance to the stationary distribution is upper bounded by $O(\ln n/\varepsilon)/(1-\lambda_2)$, where λ_2 is the second largest eigenvalue (in absolute value) of the random walk matrix. The term $1-\lambda_2$ is referred to as the spectral gap. The larger the spectral gap, the smaller the mixing time and hence the faster the random walk mixes.

A family of regular graphs with constant degree is referred to as a family of *expanders* if the spectral gap of every graph in the family is bounded away from 0. There are other "combinatorial" notions of expansion as well. A graph $G = (V, E)$ has a *vertex expansion* of α if the maximum, over all subsets S of size at most $|V|/2$, of the ratio $|N(S)|/|S|$, where $N(S)$ is the set of neighbors of vertices of S in $V - S$. We will consider other notions too, including edge expansion and conductance. For a family of constant-degree regular graphs, a graph has a positive constant spectral gap, if and only if it has a positive constant vertex expansion (and vice versa), and if and only if it has a positive constant edge expansion (and vice versa), and if and only if it has a positive constant conductance (and vice versa).

For any sufficiently large n, a random 4-regular graph over n vertices is an expander with probability at least $1/2$. This result is due to Pinkser. We will prove a weaker result for random bipartite graphs.