- Mixing time and spectral gap
- Lower bound on spectral gap
- Normalized random walk and Laplacian matrices

1 Mixing time and spectral gap

Last class we proved the following, where M is the random walk matrix for a d-regular graph.

Lemma 1. *For any initial probability distribution π, we have*

$$\|\pi M^k - u\| \leq \lambda(G)^k,$$

where

$$\lambda(G) = \max_{x \perp \mathbf{1}} \frac{\|xM\|}{\|x\|}.$$

We now take a closer look at $\lambda(G)$. For a d-regular undirected graph M is symmetric. This implies that it has n real eigenvalues and corresponding real eigenvectors. Note that u is an eigenvector with eigenvalue 1. One can see that every other eigenvalue is less than 1; and in fact, every eigenvalue of M has absolute value at most 1. (We establish both of these statements formally below.)

Let $1 = \lambda_1 \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ denote the n eigenvalues of M and let $u = v_1, v_2, \ldots, v_n$ denote the corresponding eigenvectors.

Lemma 2. *For every probability distribution π, we have*

$$\|\pi M^k - u\| \leq \lambda_2^k \|\pi - u\|.$$

Proof: We write $\pi - u$ (which is orthogonal to u) as a linear combination of the eigenvectors $v_2, \ldots, v_n$:

$$\pi - u = \sum_{i=2}^{n} c_i v_i$$

We now derive the desired inequality as follows.

$$\begin{aligned}
\|\pi M^k - u\| &= \|(\pi - u)M^k\| \\
&= \|\sum_{i=2}^{n} c_i v_i M^k\| \\
&= \|\sum_{i=2}^{n} c_i \lambda_i^k v_i\| \\
&\leq |\lambda_2|^k \|\sum_{i=2}^{n} c_i v_i\| \\
&= |\lambda_2|^k \|\pi - u\|
\end{aligned}$$

Another way to see it is to show $\lambda(G) = |\lambda_2|$, whose proof is essentially embedded above. Take any $x \perp u$. Then, it follows that $x = \sum_{i=2}^{n} a_i v_i$ for some coefficients $a_2, \ldots, a_n$. Then we have

$$\begin{aligned}
\|xM\|^2 &= \|\sum_{i=2}^{n} a_i v_i M\|^2 \\
&= \|\sum_{i=2}^{n} a_i \lambda_i v_i\|^2 \\
&\leq |\lambda_2|^2 \|\sum_{i=2}^{n} a_i v_i\|^2 \\
&= |\lambda_2|^2 \|x\|.
\end{aligned}$$

We have thus established that $\lambda(G) \leq |\lambda_2|$. To see the other direction, take $x = v_2$ and we obtain that $\|xM\|/\|x\| = |\lambda_2|$, thus showing that $\lambda(G) \geq |\lambda_2|$. □

2 Lower bound on spectral gap

Recall that we showed in the first lecture that the time it takes for the random walk to converge to the stationary distribution is inversely proportional to $1 - \lambda(G) = 1 - |\lambda_2|$, which is referred to the spectral gap (i.e., the gap between the largest eigenvalue and second largest eigenvalue, in absolute terms).

This brings us to the question: what is $|\lambda_2|$ for an arbitrary undirected graph G.

Lemma 3. *All the eigenvalues of M are at most 1 and at least -1. Furthermore, -1 is an eigenvalue for a connected graph if and only if the graph is bipartite.*

Proof: By Courant-Fischer's Theorem, the largest eigenvalue is given by

$$\max_{x \neq 0} \frac{x^T M x}{x^T x} = \max_{x \neq 0, \|x\|=1} x^T M x.$$

For any unit vector x, we have

$$\begin{aligned}
x^T M x &= \sum_i \Big(\sum_{(i,j)\in E} x_i x_j / d \Big) \\
&= \sum_{(i,j)\in E} 2 x_i x_j / d \\
&= \sum_{(i,j)\in E} (x_i^2 + x_j^2 - (x_i - x_j)^2)/d \\
&= 1 - \frac{1}{d} \sum_{(i,j)\in E} (x_i - x_j)^2 \\
&\leq 1.
\end{aligned}$$

Similarly, for any unit vector x, we have

$$\begin{aligned}
x^T M x &= \sum_i \Big(\sum_{(i,j)\in E} x_i x_j / d \Big) \\
&= \sum_{(i,j)\in E} 2 x_i x_j / d \\
&= \sum_{(i,j)\in E} ((x_i + x_j)^2 - x_i^2 - x_j^2)/d \\
&= -1 + \frac{1}{d} \sum_{(i,j)\in E} (x_i + x_j)^2 / d \\
&\geq -1.
\end{aligned}$$

Furthermore, note the equality in the preceding step happens only if for every edge (i,j), $x_i = -x_j$. This implies that the vertex set of the graph can be partitioned into two subsets — $\{i : x_i > 0\}$ and $\{j : x_j < 0\}$ — such every edge is from one subset to the other; hence the graph is bipartite. $\square$

By Courant-Fischer, the smallest eigenvalue is given by

$$\min_{x\neq 0} \frac{x^T M x}{x^T x} = 1 - \max_{x\neq 0} \frac{x^T L x}{d x^T x} = 1 - \max_{x \neq 0} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_i x_i^2} \geq 1 - \max_{x\neq 0} \frac{\sum_{(i,j)\in E} 2(x_i^2 + x_j^2)}{d \sum_i x_i^2} = -1.$$

Lemma 4. *If G is connected and nonbipartite, then $|\lambda_2|$ is at most $1 - 1/(4dn^3)$.*

Proof: Our proof is in two parts: first we show that the second largest eigenvalue is at most $1 - 1/(4dn^3)$. Second, we argue that the smallest eigenvalue is at least $-1 + 1/(4dn^3)$.

We have shown above that for a unit vector x, we have

$$x^T M x = 1 - \frac{1}{d} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Consider the second largest eigenvalue, which equals

$$\max_{x \perp \mathbf{1}, \|x\|=1} x^T M x = 1 - \min_{x\perp \mathbf{1}, \|x\|=1} \frac{1}{d} \sum_{(i,j)\in E} (x_i - x_j)^2.$$

Sort the x_i values in nondecreasing order $x_1 \le x_2 \le \ldots \le x_n$. Since $\|x\| = 1$ and $\sum_i x_i = 0$, it follows that $x_1 \le 0 \le x_n$ and $|x_1 - x_n| \ge 1/\sqrt{n}$. Therefore, there exists at least one i such that $|x_i - x_{i-1}| \ge 1/n^{3/2}$. Since the graph is connected, there exists an edge (i, j) with $|x_i - x_j| \ge 1/n^{3/2}$, which implies that the second largest eigenvalue is at most $1 - 1/(dn^3)$.

Using a similar argument, we can show that the smallest eigenvalue is at least $-1 + 1/(4dn^3)$ if the graph is nonbipartite and connected. We leave this as an exercise. □

3 Normalized random walk and Laplacian matrices

Thus far, we have focused on d-regular graphs. The associated random walk matrix in that case is symmetric and we can draw results from the spectral decomposition of such graphs to obtain bounds on the rate of convergence. What can be done in the more general setting of nonregular undirected graphs?

For a nonregular undirected graph, the adjacency matrix is still symmetric; however, the random walk matrix M, where M_{ij} equals $1/d(i)$ with $d(i)$ being the degree of i, is not. Note that $M = AD^{-1}$. Define the *normalized random walk matrix* $N = D^{-1/2} M D^{1/2}$, where D is the diagonal matrix with the ith diagonal entry being $d(i)$. We thus have

$$N = D^{-1/2} A D^{-1/2},$$

which is a symmetric matrix.

Lemma 5. *The matrices M and N have the same eigenvalues and closely related eigenvectors.*

Proof: Let v be an eigenvector of N, with eigenvalue λ. Then, we have

$$D^{-1/2} M D^{1/2} v = Nv = \lambda v.$$

Multiplying both sides by $D^{1/2}$ we obtain

$$M D^{1/2} v = \lambda D^{1/2} v.$$

Thus $D^{1/2} v$ is an eigenvector of M with eigenvalue λ. □

4 Random walks in nonregular undirected graphs

What is the stationary distribution of a random walk in an arbitrary undirected graph? It is easy to verify that the stationary distribution is given by π:

$$\pi(v) = \frac{d(v)}{\sum_{u \in V} d(u)} = \frac{d(v)}{2m},$$

where m is the number of edges. We now study the convergence rate of a random walk to the stationary distribution.

Let $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$ denote the eigenvalues of the normalized walk matrix N with corresponding eigenvectors $v_1, \ldots, v_n$. Consider probability vector p_t of the random walk at time t, with p_0 being the initial vector. Since $N = \sum_k \lambda_k v_k^T v_k$, we obtain

$$M^t = (D^{1/2} N D^{-1/2})^t = D^{1/2} N^t D^{-1/2} = \sum_{k=1}^{n} \lambda_k^t D^{1/2} v_k^T v_k D^{-1/2} = \pi + \sum_{k=2}^{n} \lambda_k^t D^{1/2} v_k^T v_k D^{-1/2}.$$

Suppose we start the random walk from vertex i, then p_0 has its ith component as 1 and the others being zero. Then, we obtain

$$p_t(j) = \pi(j) + \sum_{k=2}^{n} \lambda_k^t v_k(i) v_k(j) \sqrt{\frac{d(j)}{d(i)}}.$$

We now can bound $p_t(j)$ as follows.

$$\begin{aligned} |p_t(j) - \pi(j)| &= \left| \sum_{k=2}^{n} \lambda_k^t v_k(i) v_k(j) \sqrt{\frac{d(j)}{d(i)}} \right| \\ &\leq |\lambda_2|^t \left| \sum_{k=2}^{n} v_k(i) v_k(j) \sqrt{\frac{d(j)}{d(i)}} \right| \\ &\leq |\lambda_2|^t \sqrt{\frac{d(j)}{d(i)}}. \end{aligned}$$

In the last step, we use the fact that $\sum_{k=2}^{n} v_k(i) v_k(j) \leq 1$. This follows from the fact that the matrix Q formed by the eigenvectors $v_1, \ldots, v_n$ is orthonomal – so $Q^T Q = I$ – so every row and column is a unit vector.