• Overview of the SIS Model

• Ganesh-Massoulié-Towsley: Upper Bound on Epidemic Die-Out

1 Overview of the SIS Model

We studied SIR Model/Branching Process in the previous lecture, today we will introduce SIS (Susceptible-Infective-Susceptible) model. There are three components.

2 SIS Model

• Contact network through which the infection spreads.

• Infection Rate along each edge is $\beta$.

• Recovery Rate $\delta$.

Extent & Length of epidemic

We represent the system by a connected graph $G(V,E)$. Let $|V| = n$, and let the state at time $t$ be represented by a vector $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))^T$. $X_i$ is defined as follows:

$$X_i(t) = \begin{cases} 
1 & \text{if node } i \text{ is infected at time } t \\
0 & \text{otherwise}
\end{cases}$$

Assume that infected nodes $X_i$ contaminate neighbors as a Poisson process with rate $\beta$ and recover with a Poisson process with rate $\delta$. This defines a continuous-time Markov process with transition rates:

$X_i: 0 \rightarrow 1$ at rate $\beta \sum_{(i,j) \in E} X_j$,

$X_i: 1 \rightarrow 0$ at rate $\delta$. 
Without loss of generality, we can assume $\delta = 1$, since it essentially corresponds to a normalization factor for the die-out time.

In other words, given $X_i(t) = 0$ we have $X_i(t + dt) = 1$ with probability $[\beta \sum_{(i,j)\in E} X_j(t)]dt$ for $dt \to 0$.

**Note.** $X_i$ changes from 1 to 0 at rate $r$ means if it takes time $y$ to go from 1 to 0, then $Pr(y \geq t) = e^{-tr}$ (an exponential distribution with rate $r$).

**DIE-OUT of epidemic**

Let $\tau$ be the time takes for epidemic to die. Now the question we are interested in is: What is $E(\tau)$, as a function of $\beta$ and $G$?

### 3 Ganesh-Massoulié-Towsley: Upper Bound on Die-Out Time

Let $A$ be the adjacency matrix of $G$, $\lambda_1(A)$ be the largest eigenvalue of $A$.

**Theorem 1.** If $\beta < \frac{1}{\lambda_1(A)}$, then $E(\tau) = O(\log n)$.

**Proof:** Given $\beta \lambda_1(A) < 1$, at what $t$ does $\sum_i X_i(t) = 0$? To approach this, we consider the function $Pr(\sum_i X_i(t) > 0)$ and see when this function goes to 0. Recall that,

If $X_i(t) = 1$, then

$$X_i(t + dt) = 0 \quad \text{with probability } dt$$
$$X_i(t + dt) = 1 \quad \text{with probability } 1 - dt$$

If $X_i(t) = 0$, then

$$X_i(t + dt) = 1 \quad \text{with probability } dt \cdot \beta \cdot \sum_{(i,j)\in E} X_j(t)$$
$$X_i(t + dt) = 0 \quad \text{with probability } 1 - dt \cdot \beta \cdot \sum_{(i,j)\in E} X_j(t)$$

The transition of $X_i$ depends on the value of $X_i$, it is very hard to handle. Now we consider the continuous-time Markov process $Y = \{Y_i\}_{i \in V}$. 
\( Y_i(0) = X_i(0) \). For \( k > 0 \), the transition rates

\[
Y_i : k \rightarrow k + 1 \quad \text{at rate } \beta \sum_{(i,j) \in E} Y_j
\]

\[
Y_i : k \rightarrow k - 1 \quad \text{at rate } Y_i
\]

It is easy to see that \( Y_i \in \{0, 1, 2, \ldots \} \) (compared to \( X_i \in \{0, 1\} \)) and, when starting from the same initial conditions, \( Y_i \) stochastically dominates \( X_i \).

This stochastic dominance is obtained by a coupling argument in which we couple the elementary events in the probability space of \( X_i \) with those in the probability space of \( Y_i \). We omit the details here.

By stochastic dominance we have \( Pr(Y_i(t) \geq y) \geq Pr(X_i(t) \geq y) \). Therefore, we have \( Pr(\text{Epidemic not die-out at time } t) = Pr(\sum X_i(t) > 0) \leq Pr(\sum Y_i(t) > 0) \).

We have

\[
Y_i(t + dt) = \begin{cases} 
Y_i(t) + 1 & \text{with probability } \beta \cdot \sum_{(i,j) \in E} Y_j(t) \\
Y_i(t) - 1 & \text{with probability } Y_i(t)dt \\
Y_i(t) & \text{otherwise}
\end{cases}
\]

**Note.** In the above calculation we are ignoring the simultaneous occurrence of more than one event. This is because for \( dt \to 0 \), the associated probabilities are lower order terms (they are super-linear in \( dt \)).

Through continuous time \( \{Y_i(t + dt) = Y_i(t) + 1\} \) and \( \{Y_i(t + dt) = Y_i(t) - 1\} \) can never happen at the same time.

Since for all \( i \) we have

\[
E[Y_i(t + dt) - Y_i(t)] = (\beta \sum_{(i,j) \in E} E[Y_j] - E[Y_i])dt
\]

The transition rates for process \( Y(t) \) are such that

\[
\frac{d(E(Y(t))}{dt} = (\beta A - I)E(Y(t))
\]

where \( I \) denotes the identity matrix. Hence,

\[
E(Y(t)) = e^{Mt} \cdot Y(0) \quad \text{where } M = \beta A - I.
\]
Note. \( e^M = I + M + \frac{M^2}{2!} + \cdots \)

Consider the following fact that if \( A \) has eigenvalue \( \lambda \) then \( M = \beta A - I \) has eigenvalue \( \beta \lambda - 1 \) and \( e^M \) has eigenvalue \( e^{\beta \lambda - 1} \). We obtain
\[
\| E(Y(t)) \|_2 \leq e^{(\beta \lambda - 1)t} \cdot \| E(Y(0)) \|_2.
\]

Note. Cauchy-Schwarz inequality says that
\[
|\langle x, y \rangle| \leq \| x \|_2 \cdot \| y \|_2
\]
We obtain:
\[
\sum_{i \in V} E(Y_i(t)) \leq \| E(Y(t)) \|_2 \cdot \| 1 \|_2
\]
where \( 1 \) denotes the vector of ones, so \( \| 1 \|_2 = \sqrt{n} \).

Note. \( Pr(Y_i > 0) \geq Pr(X_i > 0) \), which implies that \( Pr(\sum Y_i > 0) \geq Pr(\sum X_i > 0) \). Moreover, by Markov Inequality it holds that:
\[
Pr(\sum Y_i > 0) = Pr(\sum Y_i \geq 1) \\
\leq \sum E(Y_i).
\]
We have
\[
Pr(\sum X_i(t) > 0) \leq Pr(\sum Y_i(t) > 0) \\
\leq \sum E(Y_i(t)) \\
\leq \| E(Y(t)) \|_2 \cdot \sqrt{n} \\
\leq \sqrt{n} e^{(\beta \lambda - 1)t} \cdot \| E(Y(0)) \|_2 \\
\leq n \cdot e^{(\beta \lambda - 1)t}
\]

Note. \( Y_i(0) = X_i(0) \), which takes values in \( \{0, 1\} \). So, \( \| E(Y(0)) \|_2 \leq \sqrt{n} \).

Say \( t_1 = \frac{100 \ln n}{1 - \beta \lambda t} \), then we have
\[
Pr(\text{epidemic not die-out at time } t_1) = Pr(\sum X_i(t_1) > 0) \leq \frac{1}{n^{99}}
\]
We have for any \( t \geq t_1 \),
\[
Pr(\text{epidemic not die-out at time } t) \leq \frac{1}{n^{99}}.
\]
since the epidemic die-out is an absorbing state in the Markov process.

Therefore,

\[
E(\tau) = \int_0^\infty Pr(\tau > t) dt
\]
\[
= \int_0^\infty Pr(\sum X_i(t) > 0) dt
\]
\[
\leq t_1 + \int_{t_1}^\infty Pr(\sum X_i(t) > 0) dt
\]
\[
\leq \frac{100 \ln n}{1 - \beta \lambda_1} + \int_{t_1}^\infty \frac{1}{n^{99}} dt
\]
\[
= O(\ln n).
\]

\[\square\]

**Theorem 2.** If

\[
\beta > \frac{c}{\lambda_1(A) - \lambda_2(A)}
\]

(where \(c > 0\) is a sufficiently large constant)

then \(E(\tau) = \Omega(\exp(n))\).