Epidemic Die-out in SIS model

1 Lower Bound on the Epidemic Die-out in SIS Model

- Along any edge infection rate $\beta$
- Healing and recovery rate 1
- Initial start state, how long before epidemic completely dies out(i.e. no infected node)?

Let $\tau$ be the time takes for epidemic to die. Now the question we are interested in is: What is $E(\tau)$, as a function of $\beta$ and $G$?

**Theorem 1.** If $\beta < \frac{1}{\lambda_1(A)}$, then $E(\tau) = O(\log n)$.

We proved this theorem in the previous lecture. Now we provide a condition under which an epidemic will survive for a long time.

**Edge expansion**

Let

$$\eta = \min_{0<|S| \leq \frac{n}{2}} \frac{E(S, \overline{S})}{|S|}$$

In the above, $E(S, \overline{S})$ denotes the number of edges connecting the set of vertices $S$ to the complementary set, $\overline{S}$. For d-regular graph, the edge expander: $\eta = \Theta(1)$, $d = \Theta(1)$, $\lambda_1(A) = \Theta(1)$

**Theorem 2.** If

$$\beta > \frac{1}{\eta}, \quad \text{then } E(\tau) = \Omega\left(\frac{(\eta \beta)^{\frac{n}{2}}}{n}\right).$$
**Proof:** Let’s look at the set $S$ of infected nodes. There are no less than $\eta|S|$ edges connecting $S$ to $\overline{S}$.

In time $dt$, we have the probability of infection through edges is $\beta E(S, \overline{S}) dt$, which is at least $\eta \beta |S| dt$. The expected number of healing nodes is $|S| dt$.

Let $X(t)$ be the number of infected nodes at time $t$. $X = |S|$. $X$ is a Markov process starting from $X(0) = 1$ with transition rates:

$X : X \rightarrow X + 1$ at rate $\beta E(S, \overline{S})$

$X : X \rightarrow X - 1$ at rate $X$

Again, $X$ is not easy to handle, we now define a Markov process $Z$ such that $Z(0)=X(0)$ and transition rates:

$Z : Z \rightarrow Z + 1$ at rate $\eta \beta Z \ (0 \leq Z < \frac{n}{2})$,

$Z : Z \rightarrow Z - 1$ at rate $Z$

Then $X$ stochastically dominates $Z$ (i.e. $Pr(X \geq x) \geq Pr(Z \geq x)$ for all $x$) since $E(S, \overline{S}) \geq \eta X \geq \eta Z$.

Now we look at how $Z$ changes. $Z \in \{0, 1, 2, \ldots, \frac{n}{2}\}$ is indeed a random walk on a line.

Focusing on an intermediate node $k$, we have

$Pr(Z : k \rightarrow k - 1) = \frac{1}{1+\eta \beta}$

$Pr(Z : k \rightarrow k + 1) = \frac{\eta \beta}{1+\eta \beta}$

The random walk starts from $Z=1$. Let $\frac{\eta \beta}{1+\eta \beta} = p$, and $\frac{1}{1+\eta \beta} = q$.

Let $q_i = Pr(\text{walk } Z \text{ starting from } i \text{ hits } n/2 \text{ before } 0)$. Then we have $q_0 =$
0, \( q^n = 1 \) and for \( i \in \{1, 2, \ldots, \frac{n}{2} - 1\} \) we obtain

\[
q_i = q \cdot q_{i-1} + p \cdot q_{i+1}
\]
\[
q \cdot q_i - q \cdot q_{i-1} = p \cdot q_{i+1} - pq_i
\]
\[
q \cdot (q_i - q_{i-1}) = p \cdot (q_{i+1} - q_i)
\]
\[
q_{i+1} - q_i = \frac{q}{p} (q_i - q_{i-1})
\]

Suppose \( q_1 = \alpha \) and \( \frac{q}{p} = r \), we have

\[
q_1 = \alpha
\]
\[
q_2 - q_1 = r\alpha
\]
\[
q_{i+1} - q_i = r^i\alpha
\]
\[
q_i = \frac{1 - r^i}{1 - r} \cdot \alpha
\]
\[
q_{\frac{n}{2}} = \frac{1 - r^\frac{n}{2}}{1 - r} \cdot \alpha = 1
\]

Therefore,

\[
q_i = \frac{1 - r^i}{1 - r} \cdot \frac{1 - r}{1 - r^\frac{n}{2}} = \frac{1 - r^i}{1 - r^\frac{n}{2}}
\]

**Note.** This is well-known as the Gambler’s Ruin problem.

Therefore,

\[
q_1 = \frac{1 - \frac{\eta}{\eta_2}}{1 - \left(\frac{1}{\eta_3}\right)^\frac{n}{2}},
\]
\[
q_i = \frac{1 - \left(\frac{1}{\eta_3}\right)^i}{1 - \left(\frac{1}{\eta_3}\right)^\frac{n}{2}}.
\]

We have

\[
q_{\frac{n}{2} - 1} = \frac{1 - \left(\frac{1}{\eta_3}\right)^\frac{n}{2} - 1}{1 - \left(\frac{1}{\eta_3}\right)^\frac{n}{2}}
\]
\[
= 1 - \frac{\left(\frac{1}{\eta_3}\right)^\frac{n}{2} - 1 - \left(\frac{1}{\eta_3}\right)^\frac{n}{2}}{1 - \left(\frac{1}{\eta_3}\right)^\frac{n}{2}}
\]
\[
= 1 - \frac{\eta \beta - 1}{(\eta \beta)^\frac{n}{2} - 1}
\]
Note that if you have \( m \) Poisson events with rate 1, then the expected time for the first event to occur is \( 1/m \). Therefore, we have the expected time for \( Z \) goes from \( n/2 \) to \( n/2 - 1 \) is \( 2/\eta \). Moreover,

\[
E(\text{number of times walk hits } \frac{n}{2} \text{ before hitting } 0) \geq (\eta \beta)^{\frac{n}{2} - 1}
\]

Thus, \( E(\tau) = \Omega(\frac{2(\eta \beta)^{\frac{n}{2} - 1}}{n}) \). \( \square \)

Expansion and Spectral gap

When \( \eta \geq d - \lambda_2(A) \), we have if \( \beta \geq \frac{1}{d - \lambda_2(A)} \), then \( \beta \geq \frac{1}{\eta} \).

**Theorem 3.** If

\[
\beta > \frac{c}{\lambda_1(A) - \lambda_2(A)} \quad \text{(where } c > 0 \text{ is a sufficiently large constant)}
\]

then \( E(\tau) = \Omega(\exp(n)) \).