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CS 7880: Network Algorithms | Scribe: Tong Zhang

- Epidemic Die-out in SIS model

1 Lower Bound on the Epidemic Die-out in SIS Model

- Along any edge infection rate β
- Healing and recovery rate 1
- Initial start state, how long before epidemic completely dies out(i.e. no infected node)?

Let τ be the time takes for epidemic to die. Now the question we are interested in is: What is $E(\tau)$, as a function of β and G?

Theorem 1. *If* $\beta < \frac{1}{\lambda_1(A)}$, *then* $E(\tau) = O(\log n)$.

We proved this theorem in the previous lecture. Now we provide a condition under which an epidemic will survive for a long time.

<u>Edge expansion</u>

Let

$$\eta = \min_{0<|S|\leq\frac{n}{2}} \frac{E(S,\overline{S})}{|S|}$$

In the above, $E(S,\overline{S})$ denotes the number of edges connecting the set of vertices S to the complementary set, $\overline{S}$. For d-regular graph, the edge expander: $\eta = \Theta(1)$, $d = \Theta(1)$, $\lambda_1(A) = \Theta(1)$

Theorem 2. *If*

$$\beta > \frac{1}{\eta}, \qquad \textit{then } E(\tau) = \Omega\big(\frac{(\eta\beta)^{\frac{n}{2}}}{n}\big).$$

Proof: Let's look at the set S of infected nodes. There are no less than $\eta|S|$ edges connecting S to $\overline{S}$.

In time dt, we have the probability of infection through edges is $\beta E(S, \overline{S})dt$, which is at least $\eta\beta|S|dt$. The expected number of healing nodes is $|S|dt$.

Let $X(t)$ be the number of infected nodes at time t. $X = |S|$. X is a Markov process starting from $X(0) = 1$ with transition rates:

$X : X \to X + 1$ at rate $\beta E(S, \overline{S})$

$X : X \to X - 1$ at rate X

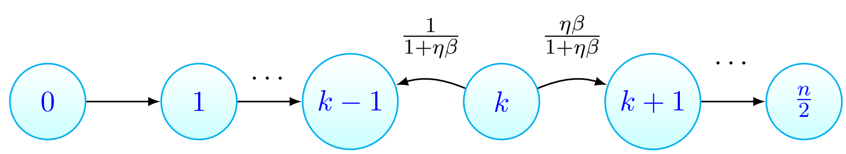
Again, X is not easy to handle, we now define a Markov process Z such that Z(0)=X(0) and transition rates:

$Z : Z \to Z + 1$ at rate $\eta\beta Z$ $\quad (0 \leq Z < \frac{n}{2})$,

$Z : Z \to Z - 1$ at rate Z

Then X stochastically dominates Z (i.e. $Pr(X \geq x) \geq Pr(Z \geq x)$ for all x) since $E(S, \overline{S}) \geq \eta X \geq \eta Z$.

Now we look at how Z changes. $Z \in \{0, 1, 2, \ldots, \frac{n}{2}\}$ is indeed a random walk on a line.



Focusing on an intermediate node k, we have

$Pr(Z : k \to k - 1) = \frac{1}{1+\eta\beta}$

$Pr(Z : k \to k + 1) = \frac{\eta\beta}{1+\eta\beta}$

The random walk starts from Z=1. Let $\frac{\eta\beta}{1+\eta\beta} = p$, and $\frac{1}{1+\eta\beta} = q$.

Let $q_i = Pr(\text{walk Z starting from i hits } n/2 \text{ before } 0)$. Then we have $q_0 =$

0, $q_{\frac{n}{2}} = 1$ and for $i \in \{1, 2, \ldots, \frac{n}{2} - 1\}$ we obtain

$$\begin{aligned} q_i &= q \cdot q_{i-1} + p \cdot q_{i+1} \\ q \cdot q_i - q \cdot q_{i-1} &= p \cdot q_{i+1} - pq_i \\ q \cdot (q_i - q_{i-1}) &= p \cdot (q_{i+1} - q_i) \\ q_{i+1} - q_i &= \frac{q}{p}(q_i - q_{i-1}) \end{aligned}$$

Suppose $q_1 = \alpha$ and $\frac{q}{p} = r$, we have

$$\begin{aligned} q_1 &= \alpha \\ q_2 - q_1 &= r\alpha \\ q_{i+1} - q_i &= r^i\alpha \\ q_i &= \frac{1 - r^i}{1 - r} \cdot \alpha \\ q_{\frac{n}{2}} &= \frac{1 - r^{\frac{n}{2}}}{1 - r} \cdot \alpha = 1 \end{aligned}$$

Therefore,

$$q_i = \frac{1 - r^i}{1 - r} \cdot \frac{1 - r}{1 - r^{\frac{n}{2}}} = \frac{1 - r^i}{1 - r^{\frac{n}{2}}}$$

Note. This is well-known as the Gambler's Ruin problem.

Therefore,

$$q_1 = \frac{1 - \frac{1}{\eta\beta}}{1 - \left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}}},$$

$$q_i = \frac{1 - \left(\frac{1}{\eta\beta}\right)^i}{1 - \left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}}}.$$

We have

$$\begin{aligned} q_{\frac{n}{2}-1} &= \frac{1 - \left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}-1}}{1 - \left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}}} \\ &= 1 - \frac{\left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}-1} - \left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}}}{1 - \left(\frac{1}{\eta\beta}\right)^{\frac{n}{2}}} \\ &= 1 - \frac{\eta\beta - 1}{(\eta\beta)^{\frac{n}{2}} - 1} \end{aligned}$$

Note that if you have m Poisson events with rate 1, then the expected time for the first event to occur is $1/m$. Therefore, we have the expected time for Z goes from $\frac{n}{2}$ to $\frac{n}{2}-1$ is $\frac{2}{n}$. Moreover,

$$E(\text{number of times walk hits } \frac{n}{2} \text{ before hitting } 0) \geq (\eta\beta)^{\frac{n}{2}-1}$$

Thus, $E(\tau) = \Omega\left(\frac{2(\eta\beta)^{\frac{n}{2}-1}}{n}\right)$. □

Expansion and Spectral gap

When $\eta \geq d - \lambda_2(A)$, we have if $\beta \geq \frac{1}{d-\lambda_2(A)}$, then $\beta \geq \frac{1}{\eta}$.

Theorem 3. *If*

$$\beta > \frac{c}{\lambda_1(A) - \lambda_2(A)} \quad \textit{(where } c > 0 \textit{ is a sufficiently large constant)}$$

then $E(\tau) = \Omega(\exp(n))$.