Introduction to random walks

Let $G$ be a $d$-regular directed graph (which is any directed graph in which the in-degree and out-degree of every vertex is $d$). Let $A$ denote its adjacency matrix, with $A_{ij} = 1$ whenever there is an edge from $i$ to $j$. A random walk on $G$ is a process that starts at time 0 at an arbitrary vertex and proceeds as follows: if the walk is at vertex $i$ at time $t$, then it is at vertex $j$ at time $t + 1$ where $j$ is chosen uniformly at random from all the out-neighbors of $i$. We can capture the random walk by the matrix $M = A/d$, which we call the random walk matrix.

A vector $x$ is referred to as a probability vector is $\sum_i x(i) = 1$. The location of the walk at time $t$ can be given by the probability vector $p_t = p_0 M^t$, where $p_0$ is the unit vector with $p_0(i) = 1$ if the walk starts at $i$ and 0 otherwise. We say that $x$ is a stationary distribution if $xM = x$. Some immediate questions are:

- Do stationary distributions always exist? It is immediate that $u = (1/n, \ldots, 1/n)$ is a stationary distribution of a random walk for every $d$-regular graph. More generally, finite Markov chain has a stationary distribution.
- Is the stationary distribution unique? In general, any finite irreducible ergodic Markov chain has a unique stationary distribution.
- Does a random walk always converge to a stationary distribution? If it does, how long does it take?

We now study the convergence of a random walk on $G$ to the stationary distribution $u$. Let $\lambda(G)$ be defined as follows:

$$\lambda(G) = \max_{x \perp u} \frac{\|xM\|}{\|x\|}.$$  

**Lemma 1.** For any initial probability distribution $\pi$, we have

$$\|\pi M^k - u\| \leq \lambda(G)^k.$$  

**Proof:** Since $\|xM\| \leq \lambda(G)\|x\|$ for any $x \perp u$, it follows that $\|(\pi - u)M\| \leq \lambda(G)\|\pi - u\|$. Since $(\pi - u)M^t \perp u$ for all $t$, it follows that $\|(\pi - u)M^t\| \leq \lambda(G)^t\|\pi - u\|$. But $uM^t = u$ and $\|\pi - u\| \leq 1$, yielding the desired claim. 

From the above lemma, it is clear that a random walk will converge to the stationary distribution if $\lambda(G) < 1$; smaller the value of $\lambda(G)$, the faster it will converge. In particular, in $\ln(n/\varepsilon)/(1 - \lambda(G))$ steps, every entry of $\pi M^k$ will be at least $(1 - \varepsilon)/n$. Is there a good upper bound on $\lambda(G)$ be? We will see that for undirected nonbipartite graphs $\lambda(G) \leq 1 - \Omega(1/n^3)$, which implies that a random walk on nonbipartite graphs converges to the uniform distribution in a polynomial number of steps.
2 Random walks in $d$-regular undirected graphs

For a $d$-regular undirected graph $M$ is symmetric. This implies that it has $n$ real eigenvalues and corresponding real eigenvectors. Note that $u$ is an eigenvector with eigenvalue 1. One can see that every other eigenvalue is less than 1; and in fact, every eigenvalue of $M$ has absolute value at most 1. Let $1 = \mu_1 \geq |\mu_2| \geq \ldots \geq |\mu_n|$ denote the $n$ eigenvalues of $M$ and let $u = v_1, v_2, \ldots, v_n$ denote the corresponding eigenvectors.

**Lemma 2.** For every probability distribution $\pi$, we have

$$\|\pi M^k - u\| \leq \mu_2^k \|\pi - u\|.$$

**Proof:** We write $\pi - u$ (which is orthogonal to $u$) as a linear combination of the eigenvectors $v_2, \ldots, v_n$:

$$\pi - u = \sum_{i=2}^{n} c_i v_i$$

We now derive the desired inequality as follows.

$$\|\pi M^k - u\| = \| (\pi - u) M^k \|$$

$$= \| \sum_{i=2}^{n} c_i v_i M^k \|$$

$$= \| \sum_{i=2}^{n} c_i \mu_i^k v_i \|$$

$$\leq |\mu_2|^k \| \sum_{i=2}^{n} c_i v_i \|$$

$$= |\mu_2|^k \|\pi - u\|.$$

Another way to see it is to show $\lambda(G) = |\mu_2|$, whose proof is essentially embedded above. Take any $x \perp u$. Then, it follows that $x = \sum_{i=2}^{n} a_i v_i$ for some coefficients $a_2, \ldots, a_n$. Then we have

$$\|x M\|^2 = \| \sum_{i=2}^{n} a_i v_i M \|^2$$

$$= \| \sum_{i=2}^{n} a_i \mu_i v_i \|^2$$

$$\leq |\mu_2|^2 \| \sum_{i=2}^{n} a_i v_i \|^2$$

$$= |\mu_2|^2 \|x\|.$$

We have thus established that $\lambda(G) \leq |\mu_2|$. To see the other direction, take $x = v_2$ and we obtain that $\|x M\|/\|x\| = |\mu_2|$, thus showing that $\lambda(G) \geq |\mu_2|$. \qed

This brings us to the question: what is $|\mu_2|$ for an arbitrary undirected graph $G$. Here are some things we can show.
1. All the eigenvalues of $M$ have absolute value at most 1. One way to see this is that $M = I - L/d$, where $L$ is the Laplacian of $G$. Thus, the eigenvalues of $M$ are $(1 - \lambda_i/d)$, where $\lambda_i$s are the eigenvalues of the Laplacian. By Courant-Fischer, the largest eigenvalue is given by

$$\max_{x \neq 0} \frac{x^T M x}{x^T x} = 1 - \min_{x \neq 0} \frac{x^T L x}{d x^T x} \leq 1.$$ 

By Courant-Fischer, the smallest eigenvalue is given by

$$\min_{x \neq 0} \frac{x^T M x}{x^T x} = 1 - \max_{x \neq 0} \frac{x^T L x}{d x^T x} = 1 - \max_{x \neq 0} \frac{\sum_{(i,j) \in E}(x_i - x_j)^2}{\sum_i x_i^2} \geq 1 - \max_{x \neq 0} \frac{\sum_{(i,j) \in E}2(x_i^2 + x_j^2)}{d \sum_i x_i^2} = -1.$$ 

2. We can also show that $M$ has an eigenvalue of $-1$ if and only if $M$ is bipartite.

3. If $G$ is connected and nonbipartite, then all eigenvalues other than the largest (which is 1) have magnitude strictly less than 1. This follows from the fact that the null-space of the Laplacian of a connected graph has dimension 1.

4. If $G$ is connected and nonbipartite, then $|\mu_2|$ is at most $1 - 1/poly(n)$. In particular, we show that $|\mu_2|$ is at most $1 - 1/(4dn^3)$. We show it in two parts: first that the second largest eigenvalue is at most $1 - 1/(4dn^3)$. Second, we argue that the smallest eigenvalue is at least $-1 + 1/(4dn^3)$.

We first make the following observation:

$$x^T M x = \sum_i \left( \sum_{(i,j) \in E} x_i x_j/d \right)$$ 

$$= \sum_{(i,j) \in E} 2x_i x_j/d$$ 

$$= \sum_{(i,j) \in E} \left( x_i^2 + x_j^2 - (x_i - x_j)^2 \right)/d$$ 

$$= 1 - \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2.$$ 

Consider the second largest eigenvalue, which equals

$$\max_{x \perp 1, ||x||=1} x^T M x = 1 - \min_{x \perp 1, ||x||=1} \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2.$$ 

Sort the $x_i$ values in nondecreasing order $x_1 \leq x_2 \leq \ldots \leq x_n$. Since $||x|| = 1$ and $\sum_i x_i = 0$, it follows that $x_1 \leq 0 \leq x_n$ and $|x_1 - x_n| \geq 1/\sqrt{n}$. Therefore, there exists at least one $i$ such that $|x_i - x_{i-1}| \geq 1/n^{3/2}$. Since the graph is connected, there exists an edge $(i,j)$ with $|x_i - x_j| \geq 1/n^{3/2}$, which implies that the second largest eigenvalue is at most $1 - 1/(dn^3)$.

We now argue that the largest eigenvalue is at least $-1 + 1/(4dn^3)$ if the graph is nonbipartite.
and connected. Note that $x^TMx$ can also be written as follows.

$$x^TMx = \sum_i \left( \sum_{(i,j) \in E} x_i x_j / d \right)$$

$$= \sum_{(i,j) \in E} 2x_i x_j / d$$

$$= \sum_{(i,j) \in E} \left( (x_i + x_j)^2 - x_i^2 - x_j^2 \right) / d$$

$$= -1 + \sum_{(i,j) \in E} (x_i + x_j)^2 / d.$$

Therefore, the largest eigenvalue equals

$$\min_{x \perp 1, \|x\|=1} x^TMx = -1 + \min_{x \perp 1, \|x\|=1} \frac{1}{d} \sum_{(i,j) \in E} (x_i + x_j)^2.$$

Without loss of generality, assume $|x_1| \geq |x_n|$, so $x_1 \leq -1/\sqrt{n}$. If there is any vertex $k$ such that $|x_k| \leq 1/(2\sqrt{n})$, then consider a shortest path from $1$ to $k$: the sum of $(x_i + x_j)^2$ along the edges $(i,j)$ of this path is at least $\sum_{(i,j) \in E} (|x_i| - |x_j|)^2$, which is at least $1/(4n^3)$. Otherwise, all vertices have $|x_k| \geq 1/(2\sqrt{n})$. Now, we consider two cases. First, there is an edge $(i,j)$ with $x_i$ and $x_j$ having the same sign. In this case $(x_i + x_j)^2 \geq 1/n$, and we are done. Otherwise, all edges are from vertices with positive $x_i$ to vertices with negative $x_i$; but this can only happen if the graph is bipartite.