Lecture Outline:

- LP Duality: Farkas Lemma and the Strong Duality Theorem
- Uncapacitated Facility Location
  - A Combinatorial "Greedy" Approach: 3-Approximation via dual fitting analysis

In this lecture, we first review LP duality, and prove the strong duality theoreom of linear programming. We then study the uncapacitated facility location problem, and present a simple greedy combinatorial algorithm. By means of a dual-fitting analysis, we show that the algorithm achieves a 3-approximation.

1 LP Duality

One can view any minimization linear program as a maximization. Consider the following linear system:

$$\begin{array}{lllll} \min & 3x_1 + 2x_2 + 8x_3 & & \\ \text{s.t.} & x_1 - x_2 + 2x_3 & \geq & 5 \\ & x_1 + 2x_2 + 4x_3 & \geq & 10 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Where Z^* is OPT, we know $Z^* = 3x_1^* + 2x_2^* + 8x_3^*$, for some $x_1^*, x_2^*, x_3^* \in P$. By adding two of the inequalities, we arrive at $2x_1 + x_2 + x_3 \geq 15$. Since $x_1^*, x_2^*, x_3^* \geq 0$, we know that $Z^* \geq 15$. But we aren't limited to addition, multiplication is another way the equations can be combined. So how is this new formulation bouned? This is done by using the dual formulation, D of the minimization, which for this problem is:

$$\begin{array}{llll} \max & 5y_1 + 10y_2 & & \\ \text{s.t.} & y_1 + y_2 & \leq & 3 \\ & -y_1 + 2y_2 & \leq & 2 \\ & 2y_1 + 4y_2 & \leq & 8 \\ & y_1, y_2 & \geq & 0 \end{array}$$

The theory of LP duality (sometimes referred to as the Strong Duality Theorem) says that if the primal LP P is bounded and feasible, then the value of the primal LP equals the value of the dual LP.

1.1 Weak Duality

Weak duality makes only the claim that the value of the primal LP is at least the value of the dual LP. Consider the primal P and its dual D:

$$\begin{array}{llcl|llcl} P & & & & D & & & \\ \min & c^T x & & & \max & b^T y & & \\ \text{s.t.} & Ax & \geq & b & \text{s.t.} & A^T y & \leq & c \\ & x & \geq & 0 & & y & \geq & 0 \end{array}$$

Suppose that x^* is an optimal solution to P and y^* is an optimal solution to D. We need only show that $c^T x^* \geq b^T y^*$.

$$\begin{aligned} c^T x^* &\geq (A^T y^*)^T x^* \\ &= y^{*T} A x^* \\ b^T y^* &\leq x^{*T} A^T y^* \\ &= (y^{*T} A x^*)^T \end{aligned}$$

Noting that the last terms in the two equations are identical (since the transpose of a scalar is the scalar itself) leads to the desired conclusion.

1.2 Strong Duality

Lemma 1 (Farkas Lemma). *For any $m \times n$ real matrix A', $m \times 1$ vector b', exactly one of the following two holds:*

1. *There exists an x' such that $A'x' \geq b'$.*
2. *There exists $y' \geq 0$ such that $A'^T y' = 0$ and $b'^T y' > 0$.*

Proof. We consider the easy direction first. Suppose both the conditions hold. Then, we have the contradiction

$$0 < b'^T y' \leq x'^T A'^T y' \leq 0.$$

Suppose there is no x' such that $A'x' \geq b'$. Then, consider the convex and closed body $K = \{A'x' - s : x' \in \Re^n, s \in \Re^m, s \geq 0\}$. Since b' does not belong to this body, there is a hyperplane separating b' from K. Hence, there exists $y' \neq 0$ such that $y'^T b' > 0$ and $y'^T A'x' \leq y'^T s$ for every $x' \in \Re^n, s \in \Re^m, s \geq 0$. By setting s to 0 and considering different values of x', we can obtain $A'^T y' = 0$. □

Farkas' Lemma can be used to prove the strong duality theorem for LPs. Let the primal and dual LPs be the following.

$$\begin{array}{lll} P: & \min c^T x & \text{subject to } Ax \geq b; x \geq 0 \\ D: & \max b^T y & \text{subject to } A^T y \leq c; y \geq 0 \end{array}$$

Theorem 1. *If the primal P and dual D are both feasible, then the optimal value z^* of the primal equals the optimal value w^* of the dual.*

Proof. Let x^* and y^* denote, respectively, optimal solutions for P and D. By weak duality, $z^* \geq w^*$. We now show that $z^* \leq w^*$. The proof is by contradiction. If $z^* > w^*$, then there does not exist a y such that

$$A^T y \leq c; y \geq 0; b^T y \geq z^*.$$

We apply Farkas Lemma with the following substitutions.

$$A' = \begin{pmatrix} -A^T \\ I \\ b^T \end{pmatrix} \; b' = \begin{pmatrix} -c \\ 0 \\ z^* \end{pmatrix} \; x' = y; \; y' = \begin{pmatrix} x \\ \delta \\ \lambda \end{pmatrix}.$$

It follows from Farkas Lemma that there exists an y' of the above form such that $y' \geq 0$, $A'^T y' = 0$, and $b'^T y' > 0$. This implies that there exists an $x \geq 0$, $\lambda \geq 0$ such that $Ax = \lambda b$ and $c^T x < \lambda z^*$.

We consider two cases. First, if $\lambda = 0$, then we have found an $x \geq 0$ such that $Ax = 0$ and $c^T x < 0$. This implies that $x^* + x$ is a feasible solution with $c^T(x^* + x) < c^T x^*$, contradicting the optimality of x^*. Second, if $\lambda > 0$, we can assume by scaling that $\lambda = 1$, which implies that x satisfies $x \geq 0$, $Ax \geq b$, and $c^T x < z^*$, again contradicting the optimality of x^*. □

2 A combinatorial greedy algorithm UFL with a dual-fitting analysis

We present a purely combinatorial greedy algorithm for UFL – Algorithm 1, and show that it achieves a 3-approximation, using a dual-fitting analysis. This algorithm is due to Mettu-Plaxton; their analysis and proof of the 3-approximation is more direct.

The integer linear program for the problem is as follows:

$$\begin{array}{lll} min & \sum_{j \in F} f_j \times y_j + \sum_{i \in V} x_{ij} \times c_{ij} & \\ s.t. & x_{ij} \leq y_j & \forall i \in V, j \in F \\ & \sum_j x_{ij} \geq 1 & \forall i \in V \\ & x_{ij}, y_j \in \{0,1\} & \forall i \in V, j \in F \end{array}$$

The LP-relaxation of this program gives:

$$\begin{array}{lll} min & \sum_{j \in F} f_j \times y_j + \sum_{i \in V} x_{ij} \times c_{ij} & \\ s.t. & x_{ij} \leq y_j & \forall i \in V, j \in F \end{array}$$

Algorithm 1: Mettu-Plaxton Approach

1. **for** *each facility* $j \in F$ **do**
   define:
   $r_j = r, r > 0$ at which $\sum_{i:c_{ij} \leq r}(r - c_{ij}) = f_j$
   $B_j = \{i \in V | c_{ij} \leq r_j\}$
2. Sort r_j in the non-decreasing order. WLOG, assume $r_1 \leq r_2 \leq \cdots \leq r_m$.
3. Let $F' := \{r_1, r_2, \cdots r_m\}$ s.t. the index-set of F' represents the facilities. Call the set $I[F']$.
4. Let $j := 1, and\ X := \emptyset$.
5. **for** *facility* $j \in I[F']$ *in order* **do**
   Set $X = X \cup \{j\}$.
   let $M_j = \{r_l \in F' | c_{jl} \leq 2 \times r_l\}$.
   Set $F' = F' \setminus M_j$ with keeping the sorted order among the remaining elements as before.
   Note: $j \in M_j$ vacuously.
   Re-label the elements in the new F' starting from index 1. (new $I[F'] \subset$ old $I[F']$.)
   If $F' = \emptyset$, break.
6. Assign each client to the nearest facility in X. Let $\sigma : V \to X$ denote the mapping.
7. Output X and σ.

$$\sum_j x_{ij} \geq 1 \qquad \forall i \in V$$

$$x_{ij}, y_j \geq 0 \qquad \forall i \in V, j \in F$$

The dual program is:

$$\begin{aligned}
max \qquad & \sum_{i \in V} v_i \\
s.t. \qquad & \sum_{i \in V} w_{ij} \leq f_j && \forall j \in F \\
& v_i - w_i j \leq c_{ij} && \forall i \in V,\ j \in F \\
& w_{ij}, v_i \geq 0 && \forall i \in V,\ j \in F
\end{aligned}$$

Theorem 2. *The running time for Algorithm 1 is polynomial in the size of input.*

Proof. Left to the reader. ☐

Now, we obtain a **<u>dual solution</u>** from Algorithm 1 as follows:

1. For each client $i \in V$, facility $j \in X$, set

$$w_{ij} = \begin{cases} r_j - c_{ij} & \text{if } i \in B_j \\ 0 & \text{otherwise} \end{cases}$$

2. For each client $i \in V$, set $v_i = min_j(w_{ij} + c_{ij})$.

Checking the **feasibility** of the dual solution:

1. For each facility $j \in X$, $\sum_{i \in B_j}(r_j - c_{ij}) = f_j$.
2. For each client $i \in V$, facility $k \in X$, $v_i = min_j(w_{ij} + c_{ij}) \leq (w_{ik} + c_{ik})$.

Consider the *Total Cost* of the solution:

$$\textsc{TotalCost} = (a) \sum_{j \in X} \sum_{i \in B_j} (w_{ij} + c_{ij}) + (b) \sum_{i \notin B_j | j \in X} c_{i\sigma(i)}.$$

Notice that the clients that contribute to (a) are disjoint from the clients that contribute to (b). Now, consider any client $i \in V$. Let $v_i = w_{ik} + c_{ik}$, where $k \in F$ is some facility that minimizes the client contribution. Let $cost(i)$ denote the cost for the client i in our solution.

Theorem 3. *$\forall i \in V$, $cost(i) \leq 3 \times v_i$.*

Proof. We will prove this by considering all cases. [From Algorithm 1] For all cases, we have: $\exists p \in X$ s.t. $r_p \leq r_k$ (p could be equal to k) and $c_{pk} \leq 2 \times r_k$.

1. Suppose, $\exists j \in X$ s.t. $i \in B_j$.
   (a) If $j = p$, then $cost(i) = (r_j - c_{ij}) + c_{ij} = r_j \leq r_k$. Now, we show that $v_i \geq r_k$. There are two possibilities:
   - If $i \in B_k$, then $v_i = (r_k - c_{ik}) + c_{ik} = r_k$. Hence, $v_i \geq r_k$.
   - If $i \notin B_k$, then $v_i = c_{ik} \geq r_k$. Hence, $v_i \geq r_k$.

   In either of the two sub-cases, we see that $v_i \geq r_k$.

   (b) If $j \neq p$, then $cost(i) = (r_j - c_{ij}) + c_{ij} = r_j \leq_1 c_{ip} \leq_2 c_{ik} + c_{kp} \leq_3 c_{ik} + 2 \times r_k$.
   The first inequality is due to the following argument. If $r_j \leq r_p$, then the inequality is straightforward since $r_j \leq r_p \leq c_{ip}$. Now consider the case $r_j > r_p$; in this case, if $r_j > c_{ip}$, then $2r_j > c_{ij} + c_{ip} \geq c_{jp}$, implying that j should not be included in X. The second inequality follows from triangle inequality induced by *metric* c, and the final inequality follows from the hypothesis.
   We are left to show $3 \times v_i \geq c_{ik} + 2 \times r_k$ for case (b) to satisfy the theorem. There are two possibilities:
   - If $i \in B_k$, then $v_i = r_k \geq c_{ik}$. Therefore, $3 \times v_i = 3 \times r_k \geq c_{ik} + 2 \times r_k$.
   - If $i \notin B_k$, then $v_i = c_{ik} \geq r_k$. Therefore, $3 \times v_i = 3 \times c_{ik} \geq c_{ik} + 2 \times r_k$.

   The claim holds true for both cases.

   Hence, for both sub-cases (a) and (b), the inequality, $cost(i) \leq 3 \times v_i$ holds.

2. Suppose $i \notin B_j$ for any $j \in X$. Then $cost(i) = c_{i\sigma(i)} \leq c_{ip}$ since $p \in X$. It follows that $cost(i) \leq c_{ip} \leq 3 \times v_i$ from the similar argument as that of case 1(b).

$\square$

Now, the total cost of our solution is given by $\textsc{TotalCost} = \sum_i cost(i) \leq \sum_i (3 \times v_i) = 3 \times \sum_i v_i$. Let OPT denote the optimal primal integral solution.

Theorem 4. $\textsc{TotalCost} \leq 3 \times OPT$.

Proof. Let $cost(v, w)$ represent the dual feasible solution that we achieve from the algorithm 1. By weak-duality, we know that a feasible dual solution gives a lower bound to the optimal primal solution, which is, in turn, a lower bound for optimal primal integral solution. Hence,

$$\textsc{TotalCost} \leq 3 \times cost(v, w) \leq 3 \times LP_{Dual} \leq 3 \times LP_{Primal} \leq 3 \times OPT.$$

$\square$