POW 11–14

11. Luckiest sheep

A wolf is located at node 0 of a ring consisting of $n+1$ nodes 0, 1, ..., n, with an (undirected) edge between node i and node $(i+1) \bmod (n+1)$ for $0 \le i \le n$. Each node i, $i > 0$, has a sheep. The wolf starts a random walk on the ring from 0. In each step, it selects one of its two neighbors uniformly at random, moves there and gobbles up the sheep present there, if not already eaten. Which sheep is likely to be eaten the last? That is, for which i is the probability that the sheep at i is eaten last the maximum?

12. Computing coefficients of a polynomial given its roots

Give an $O(n \lg^2 n)$ time algorithm that takes as input numbers $r_0, r_1, \ldots, r_n$ (where duplicates are allowed) and returns the unique degree-$(n+1)$ polynomial with roots $r_0, r_1, \ldots, r_n$.

13. Ancestral depth following a splay operation

Let T be a binary search tree, let u be a leaf of T, and let v be an ancestor of u. Let T' denote the tree obtained from T by performing a splay operation at u. (Thus u is the root of T'.) Prove that if v has depth d in T (i.e., there are d edges on the path from v to the root of T), then it has depth at most $\lceil d/2 \rceil + 2$ in T'.

14. Do ranks and heights induce the same notion of balance?

Recall that we defined the *rank* $r(x)$ of a node x of a tree to be $\lfloor \log_2 n(x) \rfloor$, where $n(x)$ is the number of nodes (including x) in the subtree rooted at x. And we defined the *total rank* of the tree to be the sum of the ranks of all of the nodes in the tree.

In the analysis of splay trees, we have seen that the total rank of a complete binary tree – a perfectly balanced binary tree – with n nodes is $\Theta(n)$, while the total rank of a line – a perfectly unbalanced tree! – of n nodes is $\Theta(n \lg n)$. We also know that the height of a complete binary tree is $\Theta(\lg n)$, while that of a line is $\Theta(n)$. We now ask whether the total rank and height measures are closely related in terms of the notion of balanced trees that they induce.

Prove or disprove: If the total rank of a binary tree is $O(n)$, then the height of the tree is $O(\lg n)$.