

Problem of the Week – 1

Fitting hats on heads

Planet Red Sox Nation ordered a Red Sox baseball hat for each of their $n = 100,000$ members through IPS, the Interplanetary Parcel Service. Each hat was ordered so as to perfectly fit the head of the corresponding alien. The hats were all delivered perfectly, but they were all delivered together and no size was listed on any hat. Though all the alien heads and hats are round, there is no way to compare one hat with another (the hats are too soft) or one alien head with another (they don't like that). The only comparison possible is by fitting a hat on a head. If a hat perfectly fits a head, then it will be too big for smaller heads and too small for larger heads, and vice versa.

The leader of the planet desperately and quickly wants to determine which hat belongs to which alien, using the minimum number of hat-head fittings as possible. A deterministic procedure that needs $n(n-1)/2 = \Theta(n^2)$ fittings is easy to give. Can you help the leader do this with a *randomized* algorithm that uses expected $O(n \log n)$ fittings? To earn the planet's Medal of Honor, try coming up with an $o(n^2)$ -fitting *deterministic* algorithm.