$M/G/1$ queueing system with priorities

Lecture Outline:

- $M/G/1$ with priorities.
- Midterm.

1 $M/G/1$ with priorities

Consider the $M/G/1$ system with priorities. Each customer falls in one of k classes, numbered 1 through k. Customers in class i are given priority over customers in class j for any $j > i$. Thus, for example, class 1 has the highest priority. We assume that the priority rule is *nonpreemptive*; that is, a customer undergoing service is allowed to complete service without interruption even if a customer with a higher priority arrives in the interim. Finally, we assume that within a class, the customers are served in order of their arrival.

Let the arrival process for customers in class i be Poisson with rate λ_i. Let the service time for a customer in class i be given by the random variable X_i. Thus $\overline{X_i}$ and $\overline{X_i^2}$ are the mean service time and mean squared service time for class i. Let W_i denote the average waiting time for a customer in class i. We begin by calculating W_1. Since class 1 has the highest priority, any customer upon arrival, has to wait for the customer being served to complete and the other customers of class 1 in queue. If R denotes the average residual time and Q_i denotes the average number of customers of class i waiting in queue, we get

$$W_1 = R + Q_1\overline{X_1}.$$

Since $Q_1 = W_1\lambda_1$, we get

$$W_1 = \frac{R}{1-\rho_1},$$

where $\rho_i = \lambda_i\overline{X_i}$. Now consider class 2. A customer A in class 2 upon arrival needs to wait for the customer currently undergoing service to complete, then the customers of class 1 and 2 that are waiting in queue, and the customers of class 1 that will arrive during the period that A is waiting. The average waiting time W_2 for A thus satisfies:

$$W_2 = R + Q_1\overline{X_1} + Q_2\overline{X_2} + \lambda_1 W_2\overline{X_1}.$$

Since $Q_i = W_i\lambda_i$, we obtain

$$W_2 = R + \rho_1 W_1 + \rho_2 W_2 + \rho_1 W_2,$$

which yields

$$W_2 = \frac{R + \rho_1 W_1}{1-\rho_1-\rho_2}.$$

Plugging in the value for W_1, we get

$$W_2 = \frac{R}{(1-\rho_1)(1-\rho_1-\rho_2)}.$$

The above analysis can be generalized to derive W_i for any $i > 1$. We obtain

$$W_i = \frac{R}{(1-\rho_1-\ldots-\rho_{i-1})(1-\rho_1-\ldots-\rho_i)}.$$

It remains to calculate R. Using the graphical argument in the derivation of the P-K formula, we can show that

$$R = \frac{1}{2}\sum_{i=1}^{k} \lambda_i \overline{X_i^2},$$

This establishes that the average waiting time for a customer in class i is

$$W_i = \begin{cases} \frac{\sum_{i=1}^{k} \lambda_i^2 \overline{X_i^2}}{2(1-\rho_1)} & \text{if } i = 0, \\ \frac{\sum_{i=1}^{k} \lambda_i^2 \overline{X_i^2}}{2((1-\rho_1-\ldots-\rho_{i-1})(1-\rho_1-\ldots-\rho_i)} & \text{if } i > 0. \end{cases}$$