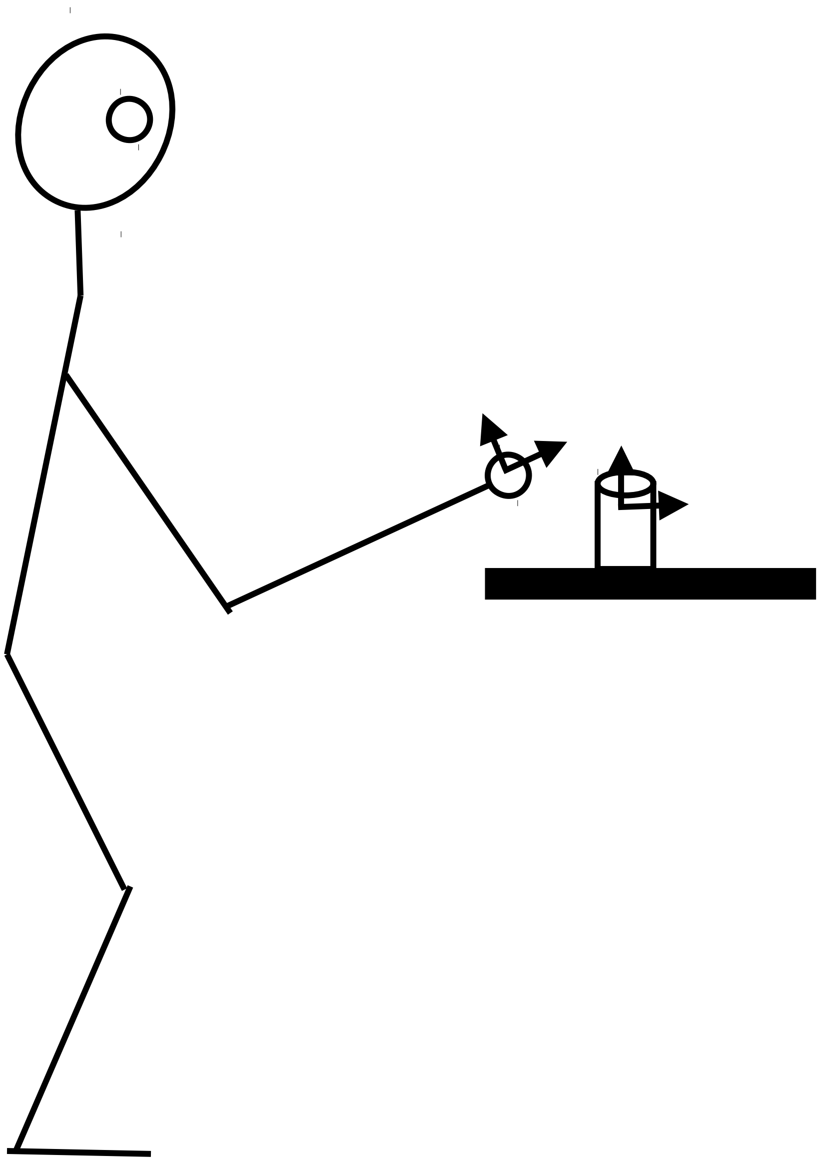
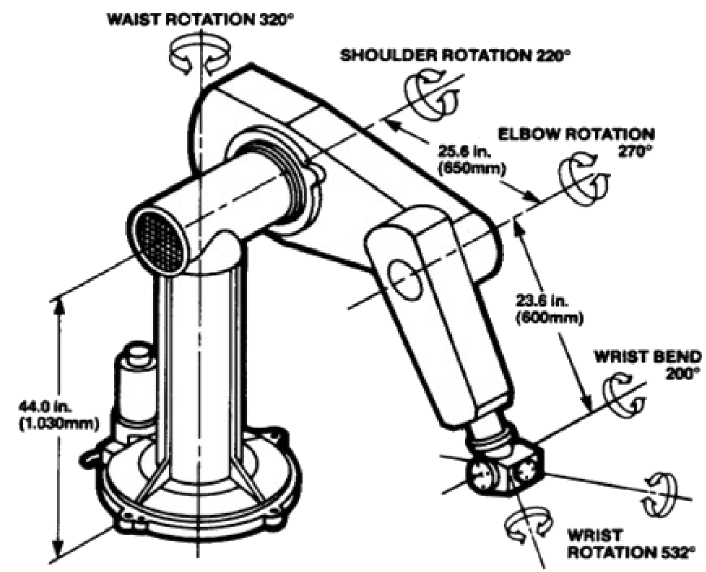
Vectors, Matrices, Rotations

Why are we studying this?



You want to put your hand on the cup…

- Suppose your eyes tell you where the mug is and its orientation in the robot *base frame* (big assumption)
- In order to put your hand on the object, you want to align the coordinate frame of your hand w/ that of the object
- This kind of problem makes representation of pose important...



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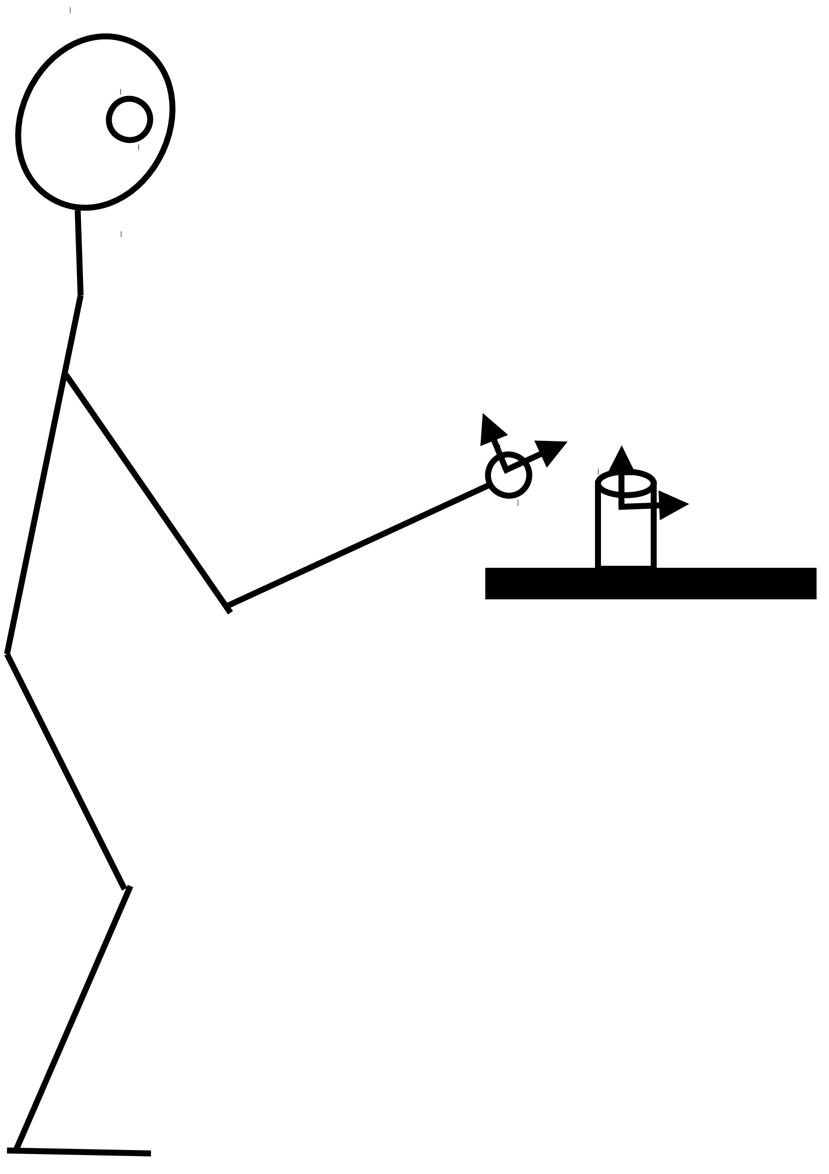
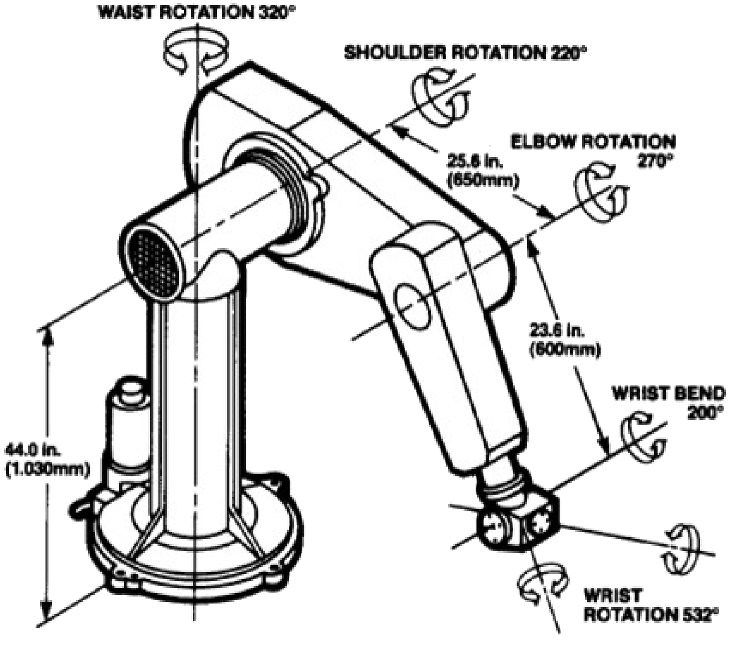
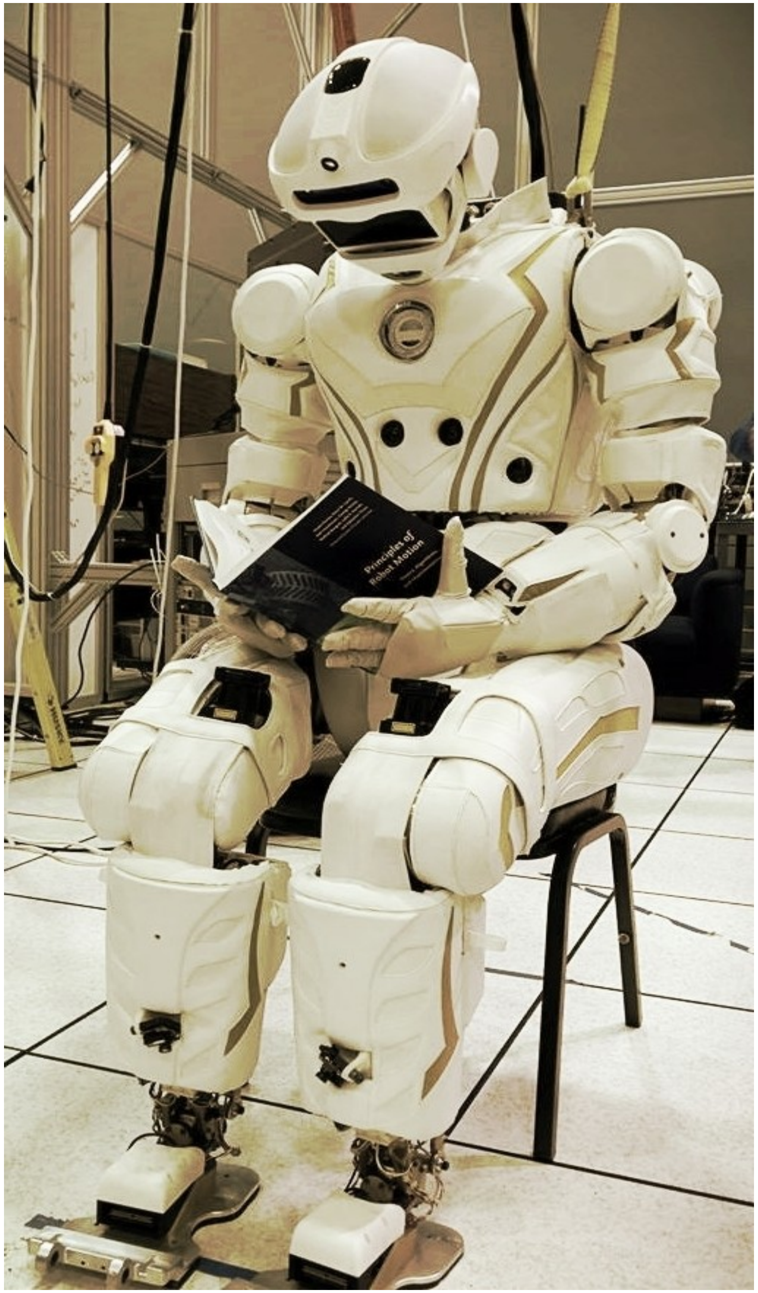




Puma 500/560

Why are we studying this?



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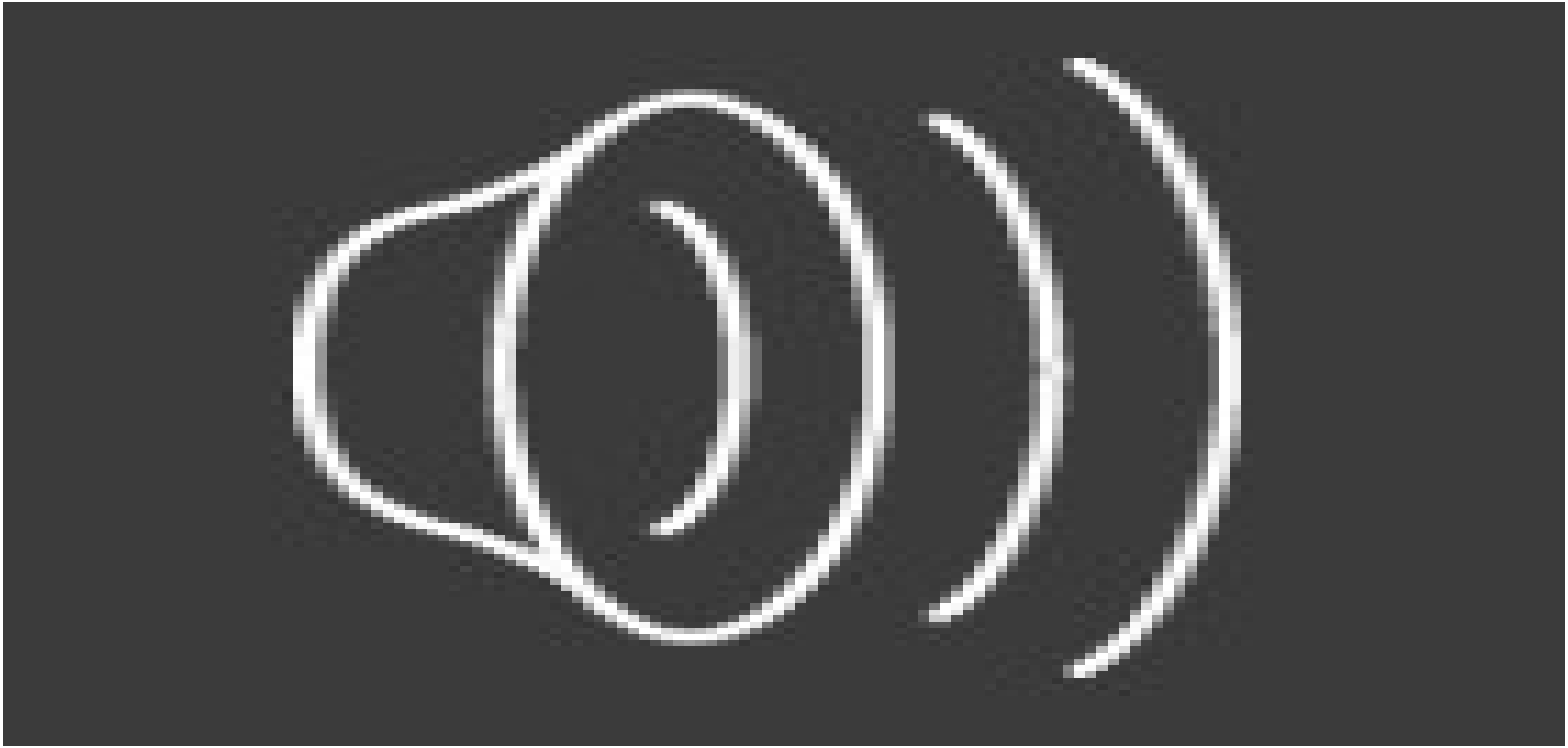
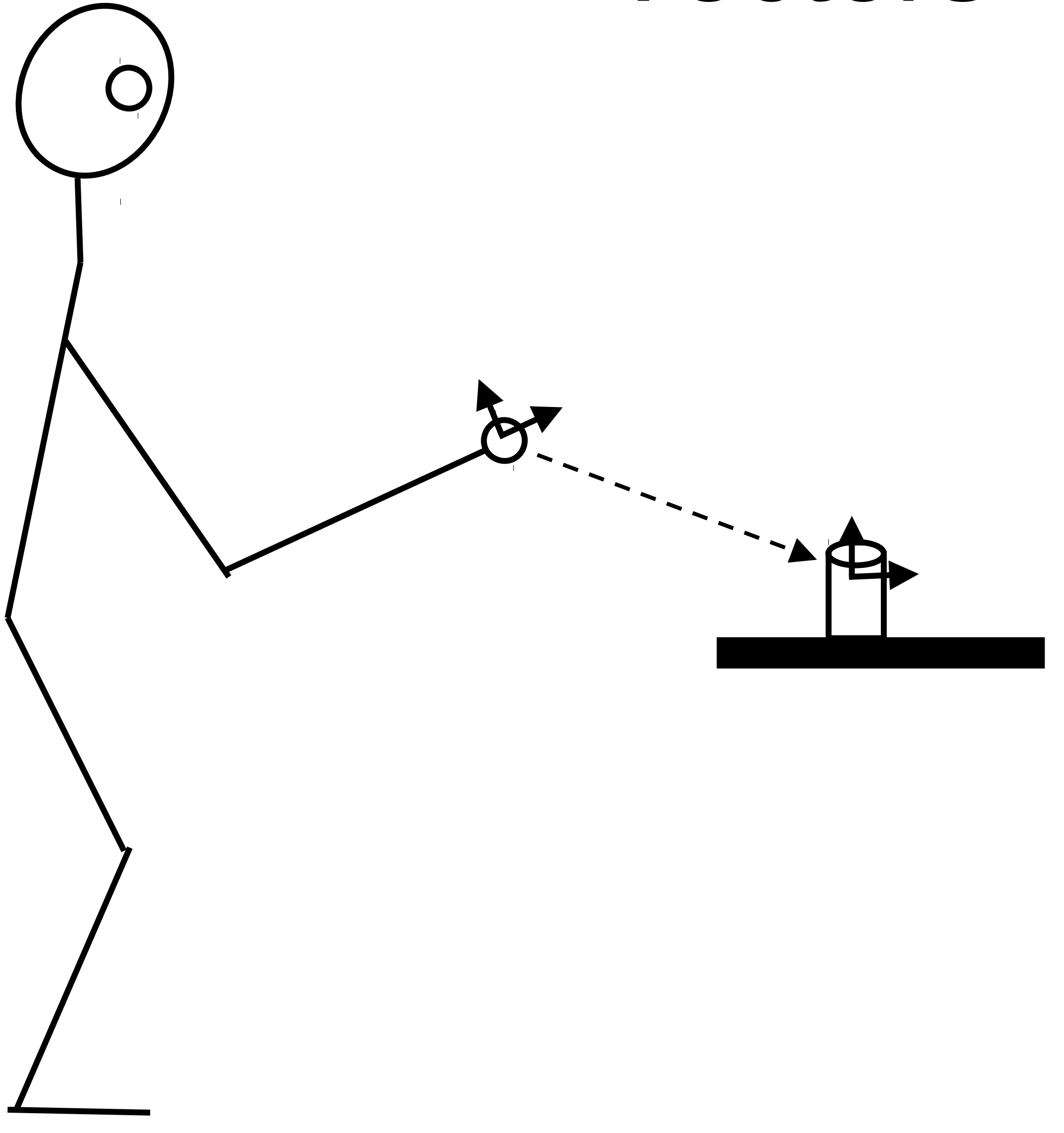


Representing Position: Vectors

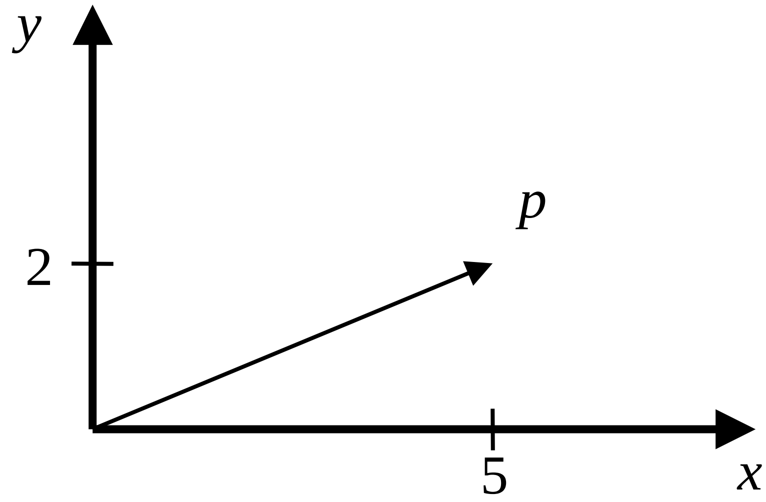


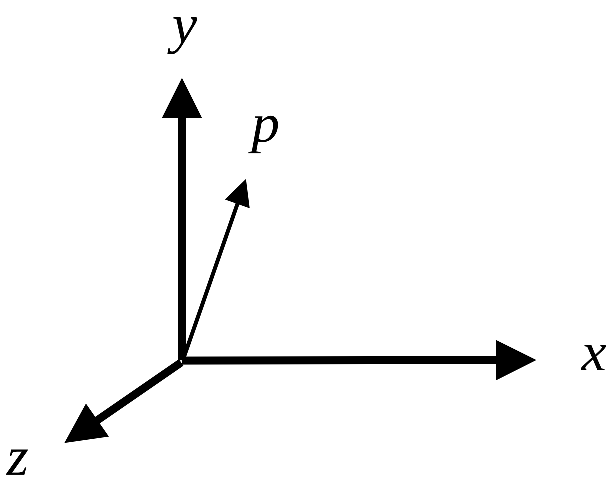
Representing Position: vectors

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$ ("column" vector)

$$p = [5, 2]$$ ("row" vector)

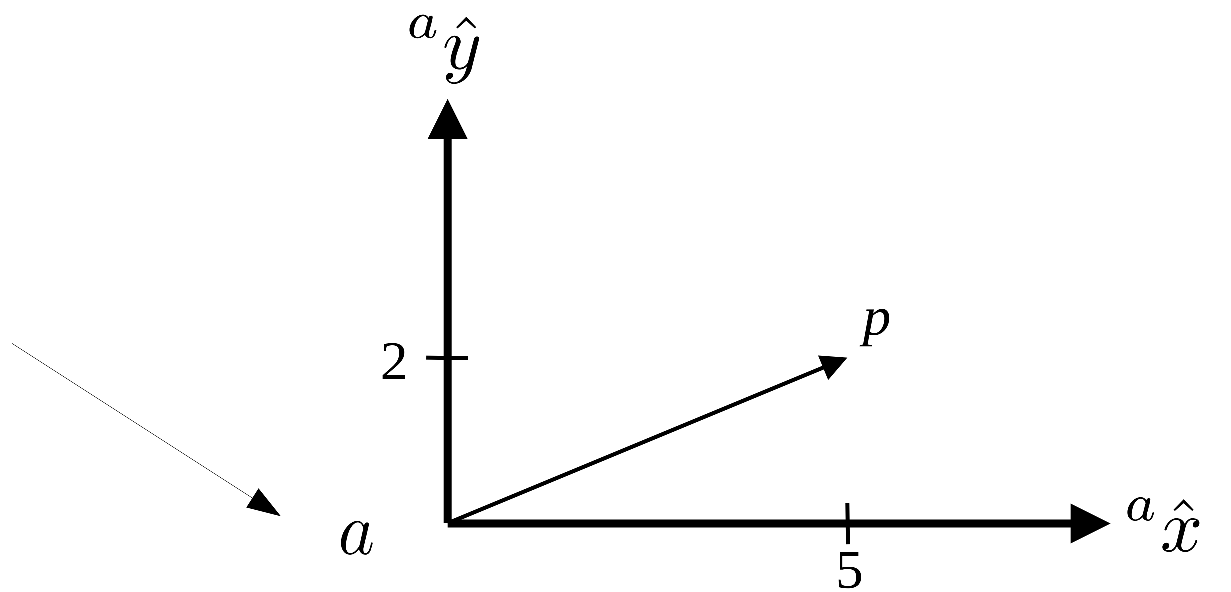
$$p = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$





Representing Position: vectors

The "a" reference frame



$({}^a\hat{x}, {}^a\hat{y})$ ← Basis vectors
– unit vectors (length of magnitude 1)
– orthogonal (perpendicular to each other)

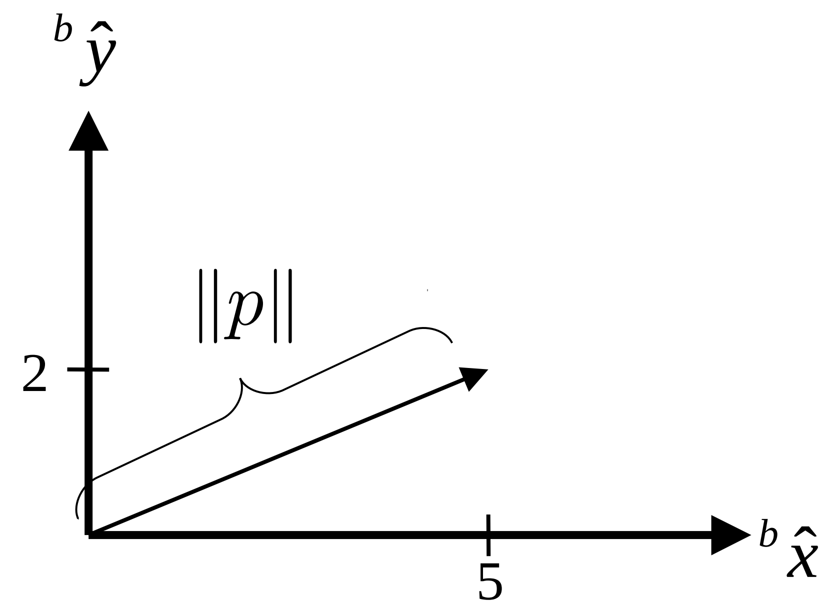
$$^a p = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$ ← Vector *p* in written in *a* reference frame

What is this unit vector you speak of?

These are the elements of *p*: $p = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$

Vector length/magnitude:

$$\|p\| = \sqrt{p_x^2 + p_y^2}$$



Definition of unit vector: $\|\hat{p}\| = 1$

You can turn an arbitrary vector *p* into a unit vector of the same direction this way: $\hat{p} = \frac{p}{\|p\|}$

And what does orthogonal mean?

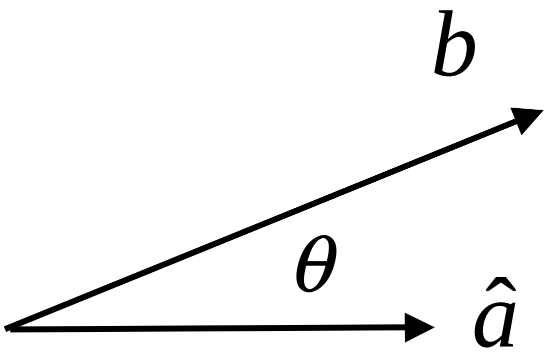
First, define the dot product: $$a \cdot b = a_x b_x + a_y b_y$$

$$= |a||b|\cos(\theta)$$

$a \cdot b = 0$ when: $a = 0$

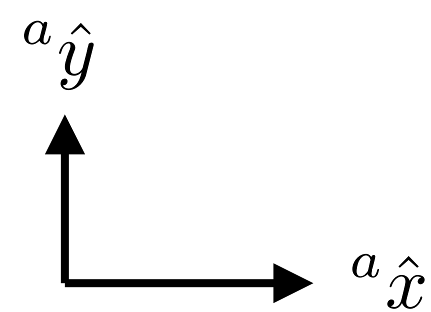
or, $b = 0$

or, $\cos(\theta) = 0$



Unit vectors are orthogonal *iff* (if and only if) the dot product is zero:

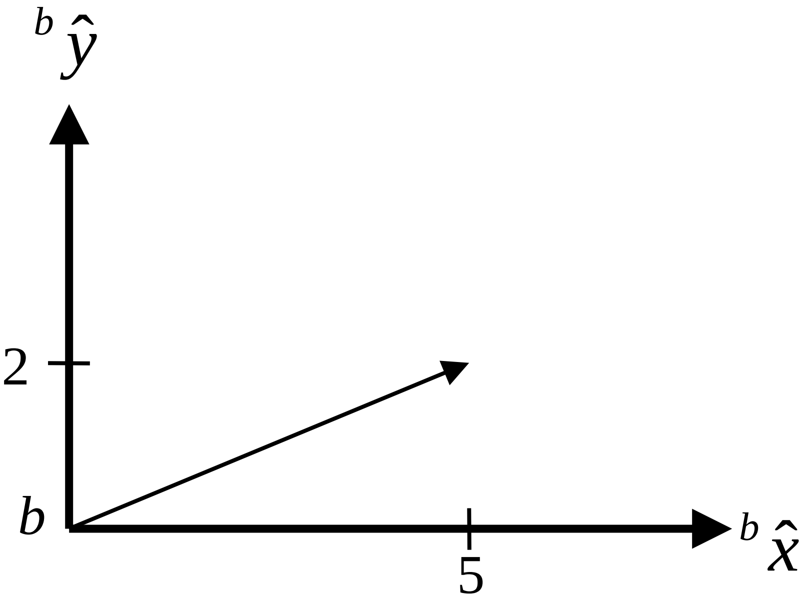
${}^{a}\hat{x}$ is orthogonal to ${}^{a}\hat{y}$ iff ${}^{a}\hat{x} \cdot {}^{a}\hat{y} = 0$

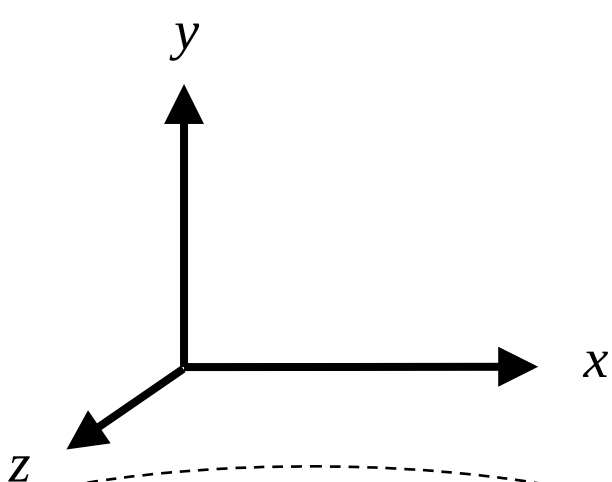


A couple of other random things

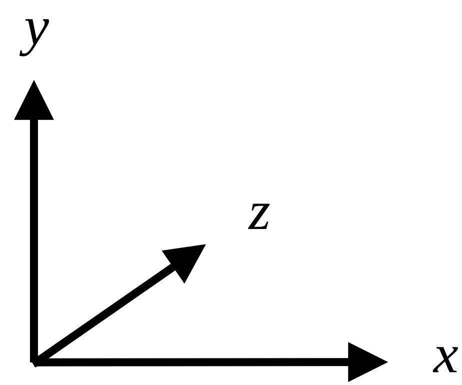
$$^{b}p = 5\ ^{b}\hat{x} + 2\ ^{b}\hat{y}$$

Vectors are elements of R^n



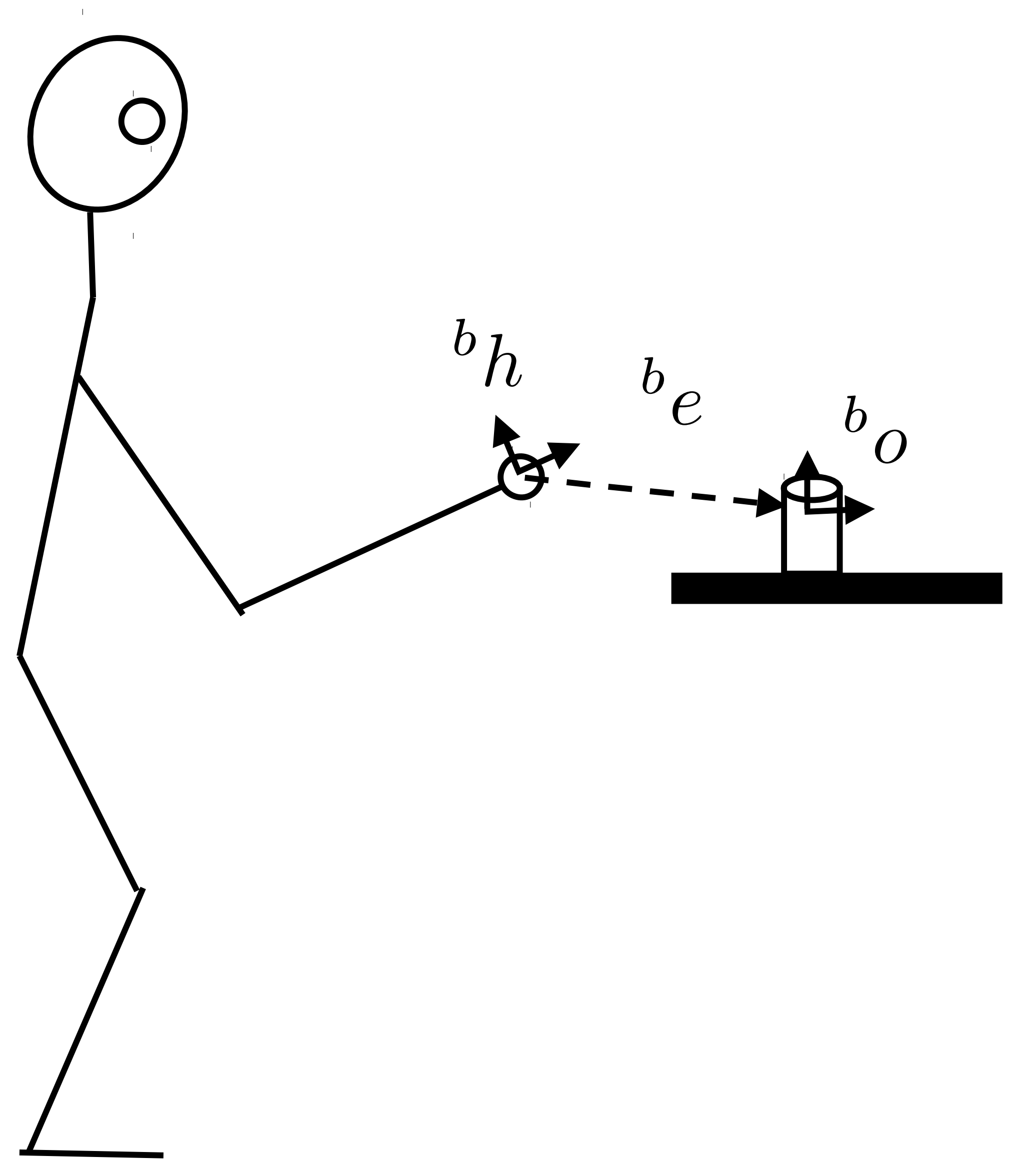


right-handed
coordinate frame



left-handed
coordinate frame

The importance of differencing two vectors



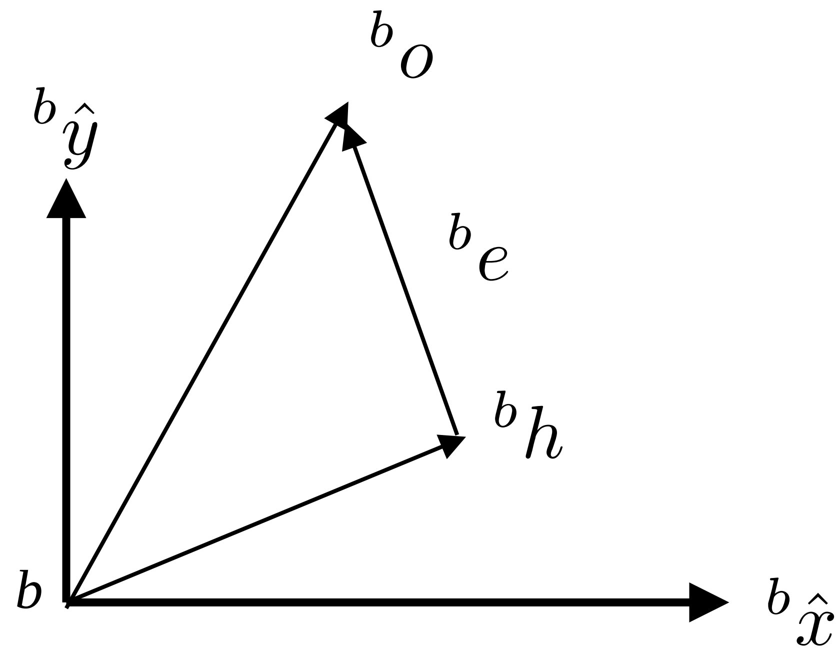
$$ {}^b o - {}^b h = {}^b e $$

The hand needs to make a Cartesian displacement of this much to reach the object

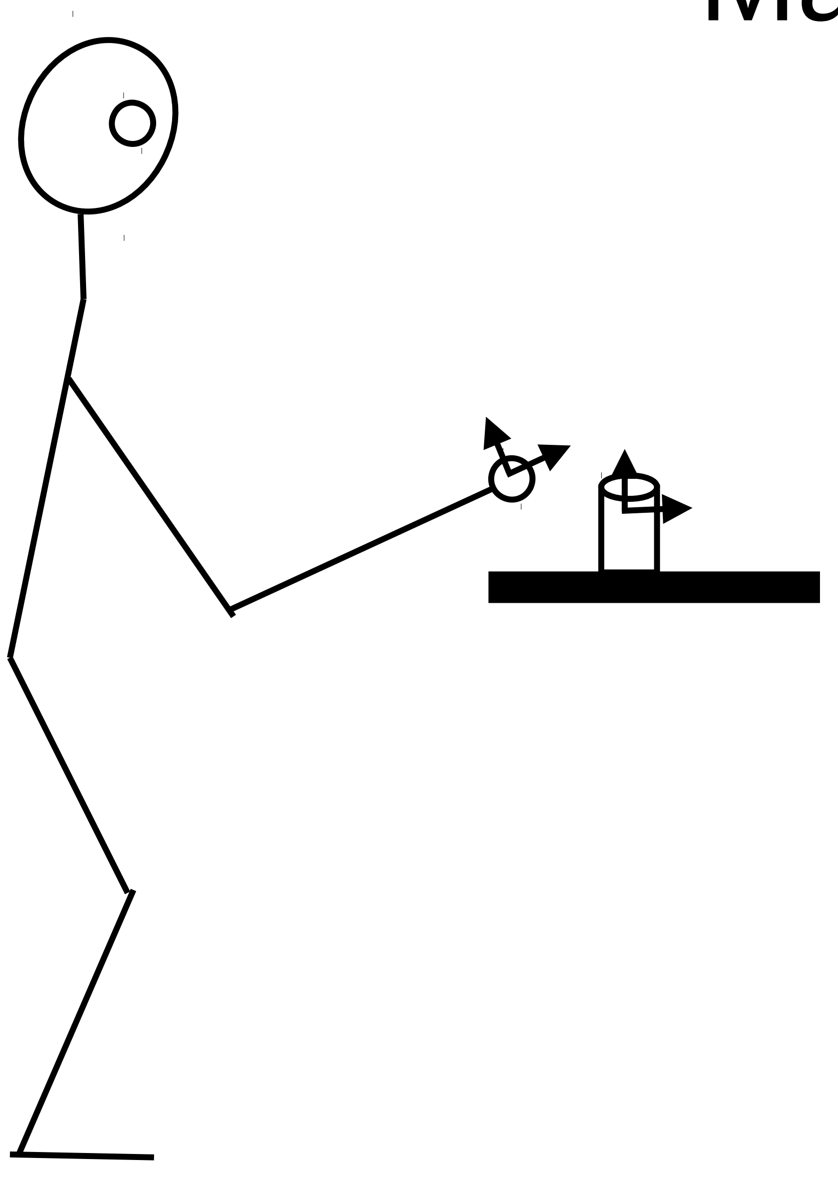
The importance of differencing two vectors

$$ {}^{b}o - {}^{b}h = {}^{b}e $$

The hand needs to make a Cartesian displacement of this much to reach the object



Representing Orientation: Rotation Matrices



- The reference frame of the hand and the object have different orientations
- We want to represent and difference orientations just like we did for positions…

Before we go there – review of matrix transpose

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \longrightarrow p^T = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

Important property: $\mathbf{A}^T \mathbf{B}^T = (\mathbf{B}\mathbf{A})^T$

and matrix multiplication…

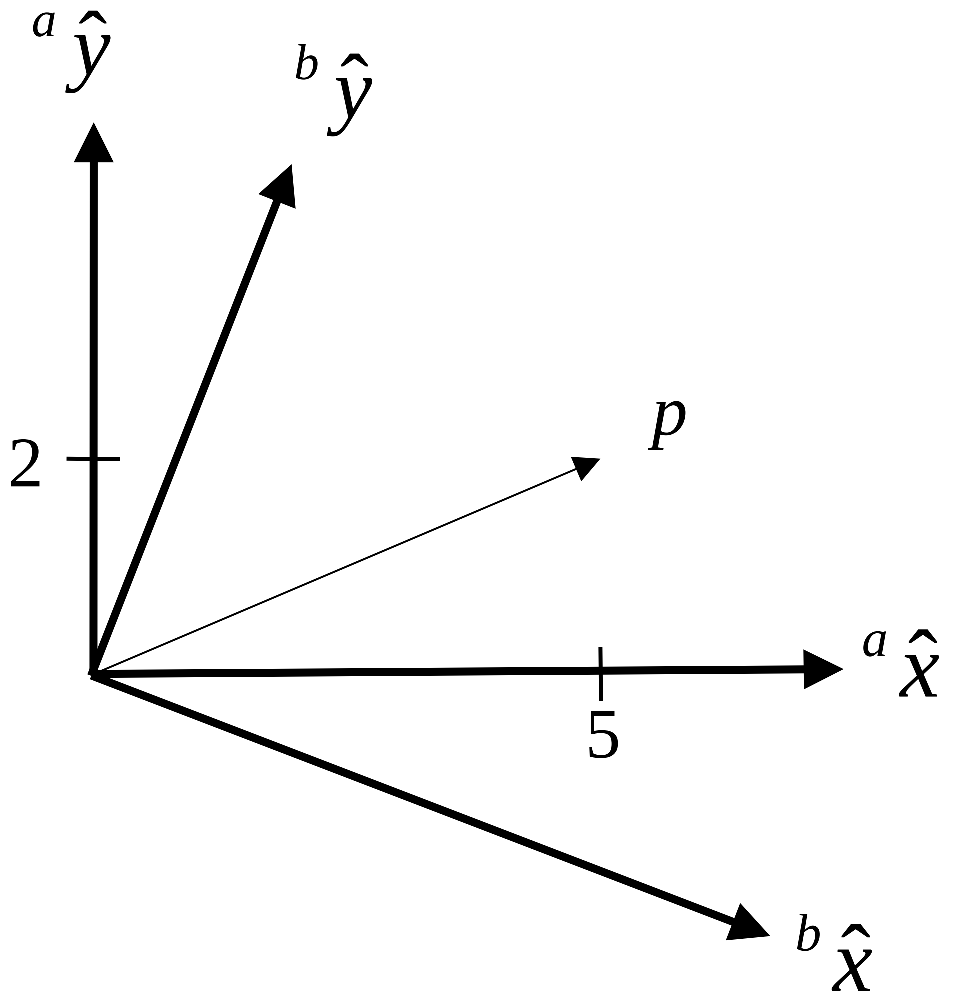
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Can represent dot product as a matrix multiply:

$$a \cdot b = a_x b_x + a_y b_y = \begin{bmatrix} a_x & a_y \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a^T b$$

Same point - different reference frames

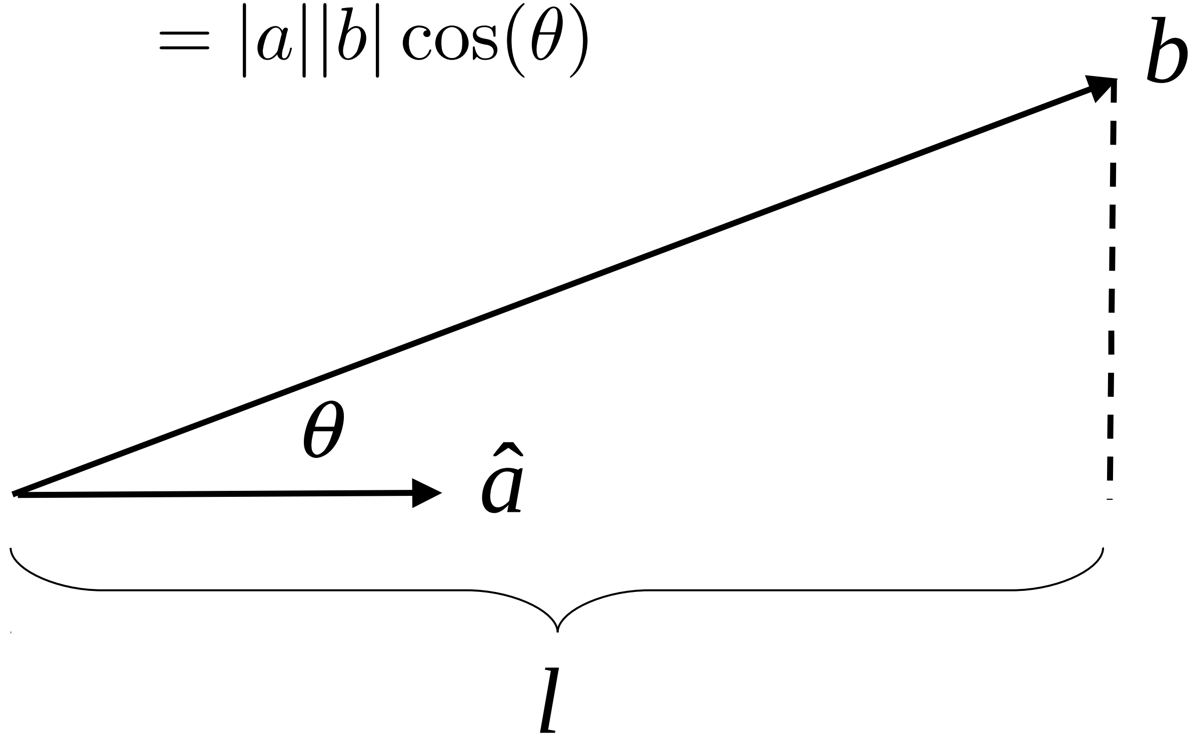


$$^a p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad ^b p = ?$$

Another important use of the dot product: projection

$$a \cdot b = a_x b_x + a_y b_y$$

$$= |a||b| \cos(\theta)$$



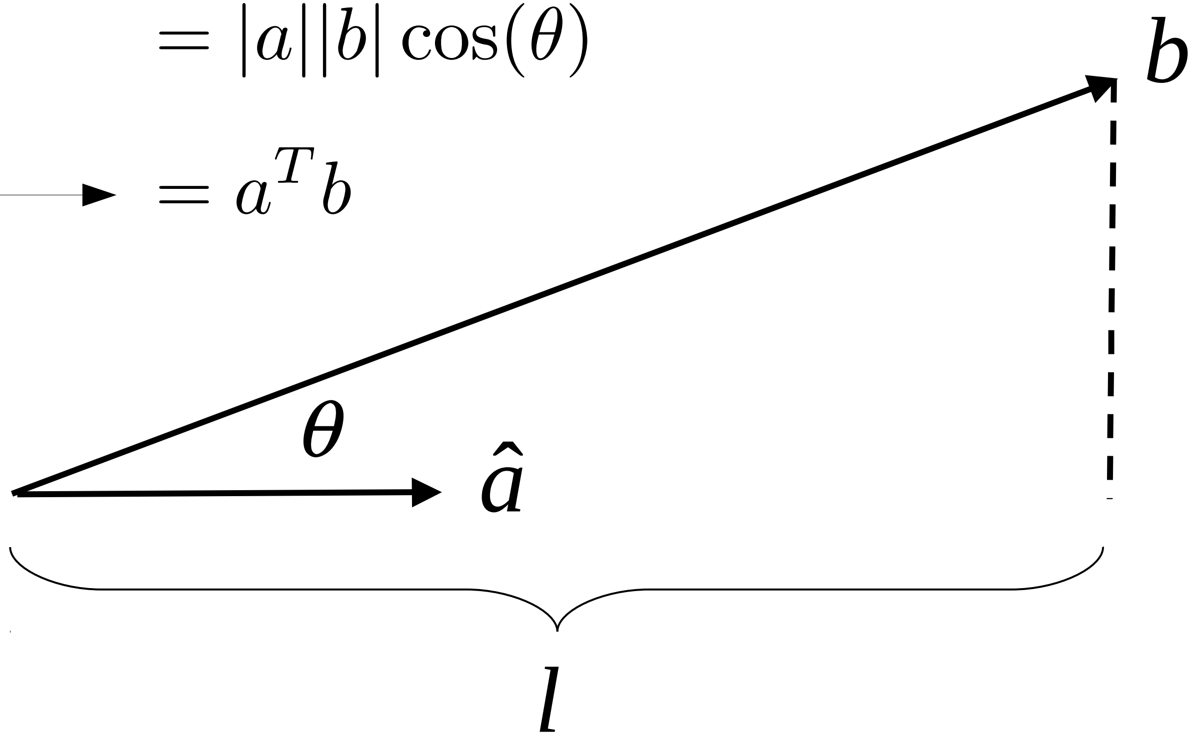
$$l = \hat{a} \cdot b = |\hat{a}||b| \cos(\theta) = |b| \cos(\theta)$$

Another important use of the dot product: projection

$$a \cdot b = a_x b_x + a_y b_y$$

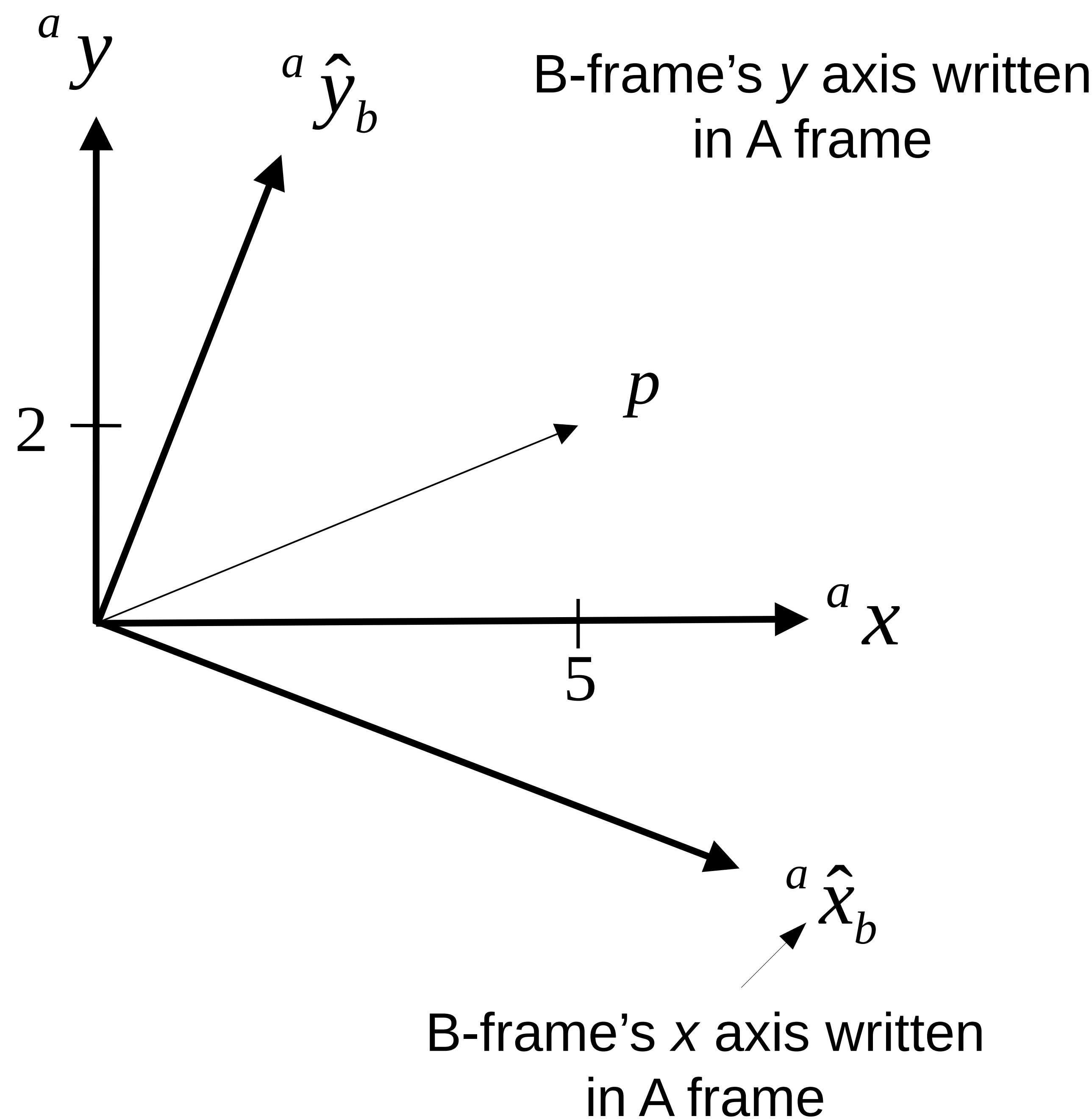
$$= |a||b|\cos(\theta)$$

Another way of writing the dot product → $= a^T b$

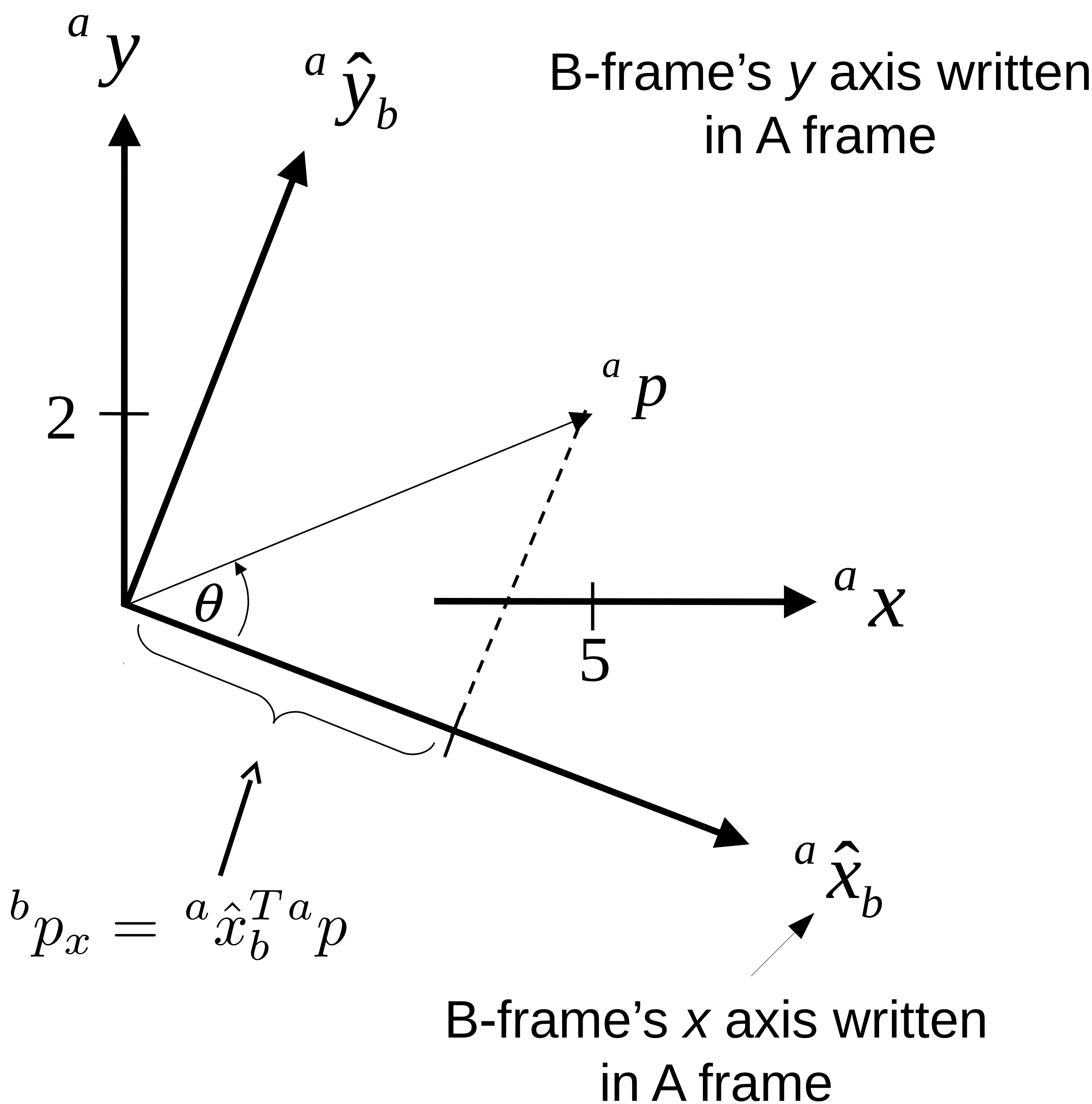


$$l = \hat{a} \cdot b = |\hat{a}||b|\cos(\theta) = |b|\cos(\theta)$$

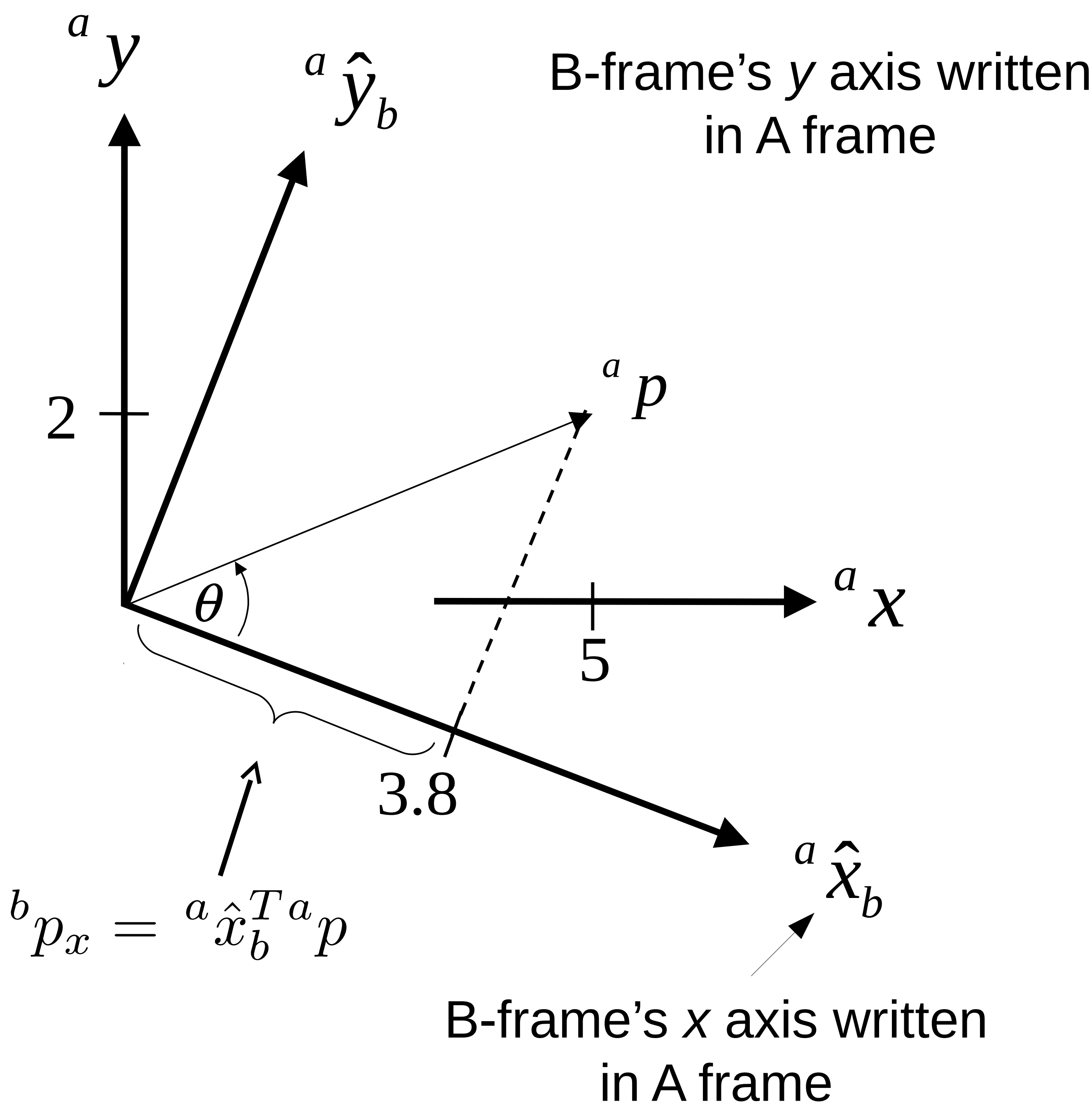
Same point - different reference frames



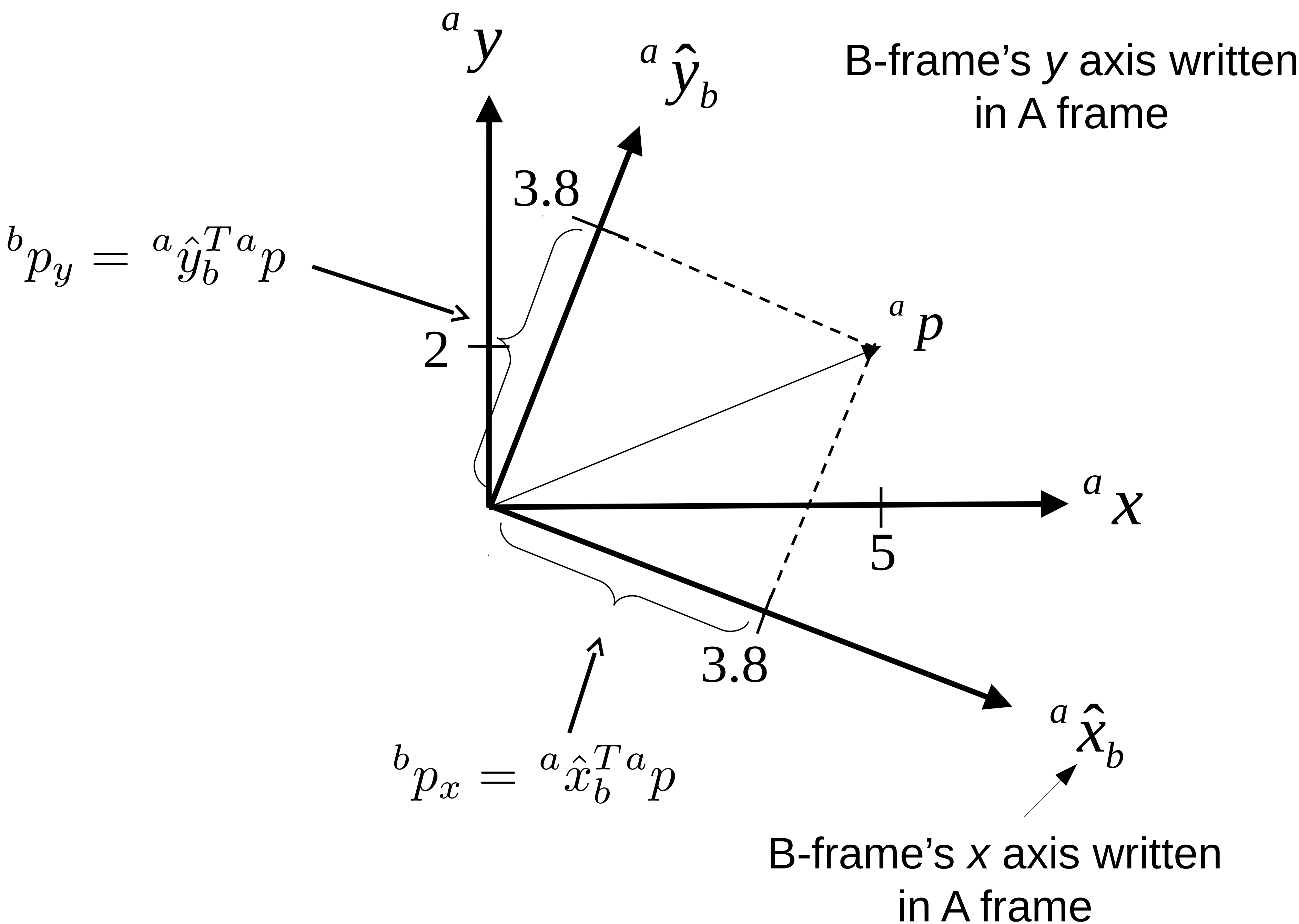
Same point - different reference frames



Same point - different reference frames



Same point - different reference frames



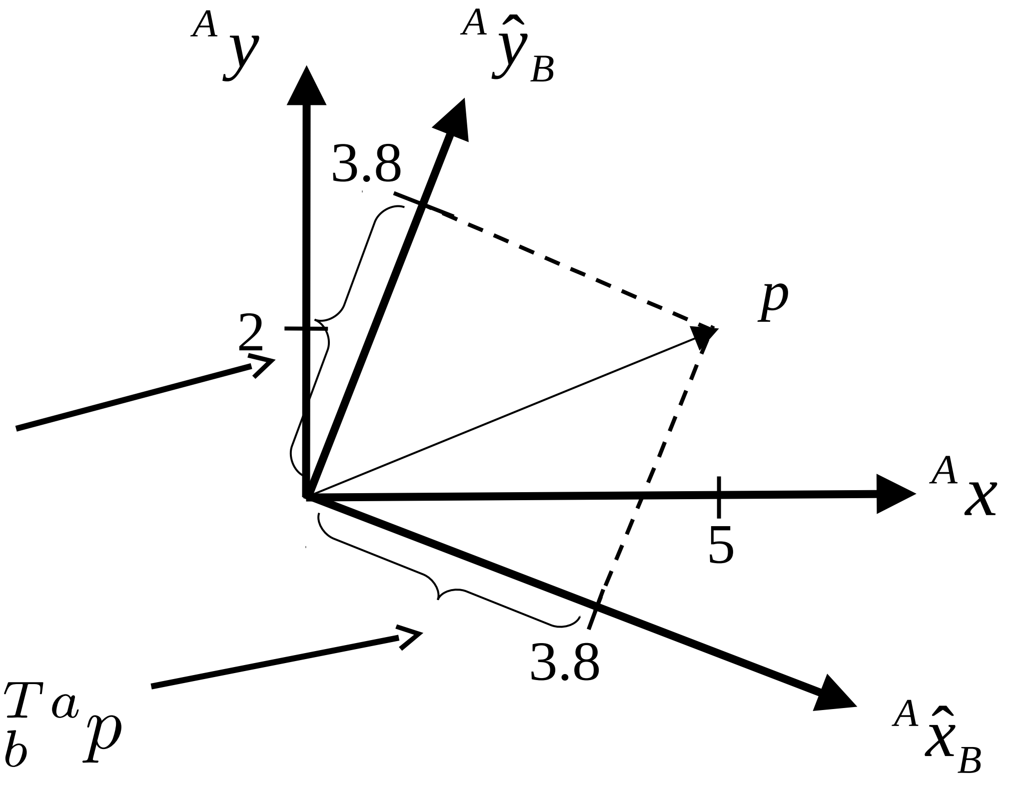
Same point - different reference frames

$$^{b}p = \begin{pmatrix} ^{a}\hat{x}_b^T \; ^{a}p \\ ^{a}\hat{y}_b^T \; ^{a}p \end{pmatrix} = \begin{pmatrix} ^{a}\hat{x}_b^T \\ ^{a}\hat{y}_b^T \end{pmatrix} {}^{a}p$$

$$= {}^{a}R_b^T \; {}^{a}p$$

where: $${}^{a}R_b^T = \begin{bmatrix} ^{a}\hat{x}_b^T \\ ^{a}\hat{y}_b^T \end{bmatrix}$$

or $${}^{a}R_b = \left[{}^{a}\hat{x}_b \; {}^{a}\hat{y}_b \right]$$



$$^{b}p_y = {}^{a}\hat{y}_b^T \, {}^{a}p$$

$$^{b}p_x = {}^{a}\hat{x}_b^T \, {}^{a}p$$

The rotation matrix

To recap: $$ {}^{b}p = {}^{a}R_b^T \; {}^{a}p $$

where: $$ {}^{a}R_b = \left[{}^{a}\hat{x}_b \; {}^{a}\hat{y}_b \right] $$

The rotation matrix

To recap: $${}^b p = {}^a R_b^T \; {}^a p$$

where: $${}^a R_b = \left[{}^a \hat{x}_b \;\; {}^a \hat{y}_b \right]$$

We will write: $${}^b R_a = {}^a R_b^T$$

so: $${}^b p = {}^b R_a \; {}^a p$$

Notice the way the notation “cancels out”

But, can we do this: $${}^b p \longrightarrow {}^a p$$???

The rotation matrix

But, can we do this: ${}^{b}p \longrightarrow {}^{a}p$???

$${}^{a}p \longrightarrow {}^{b}p \quad \Rightarrow \quad {}^{b}p = {}^{b}R_a \, {}^{a}p$$

Multiply both sides by inverse: $${}^{a}p = {}^{b}R_a^{-1} \, {}^{b}p$$

It turns out that: $${}^{b}R_a^{-1} = {}^{b}R_a^{T}$$

because the columns of ${}^{b}R_a$ are unit, orthogonal

The rotation matrix

But, can we do this: $^{b}p \longrightarrow {}^{a}p$???

$$^{a}p \longrightarrow {}^{b}p \quad \Rightarrow \quad ^{b}p = {}^{b}R_a \; ^{a}p$$

Multiply both sides by inverse: $^{a}p = {}^{b}R_a^{-1} \; ^{b}p$

It turns out that: $^{b}R_a^{-1} = {}^{b}R_a^{T}$ ⇐ This is important!

because the columns of $^{b}R_a$ are orthogonal

The rotation matrix

So, if: $${}^{b}p = {}^{b}R_{a}\ {}^{a}p$$

Then: $${}^{a}p = {}^{b}R_{a}^{T}\ {}^{b}p$$
$$= {}^{a}R_{b}\ {}^{b}p$$

The rotation matrix

Both columns are orthogonal

$$ {}^{a}R_b = ({}^{a}\hat{x}_b \ {}^{a}\hat{y}_b) $$

$$ = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} $$

But: $$ {}^{a}R_b = {}^{b}R_a^T $$

$$ = \begin{pmatrix} {}^{b}\hat{x}_a^T \\ {}^{b}\hat{y}_a^T \end{pmatrix} $$

So, the rows are orthogonal too!

$$ = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} $$

The rotation matrix

$$^{a}R_b = ({}^{a}\hat{x}_b \; {}^{a}\hat{y}_b)$$

$$= \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

Both columns are orthogonal

The same matrix can be understood both ways!

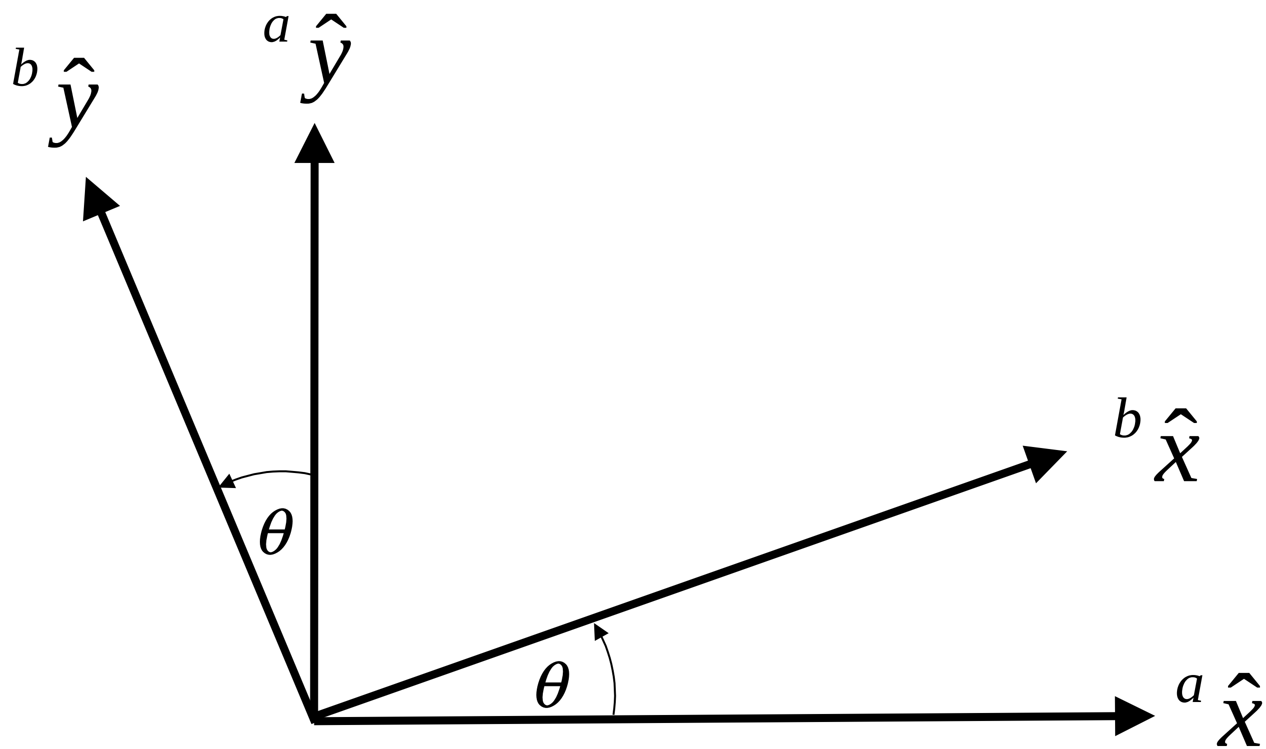
But: $$^{a}R_b = {}^{b}R_a^T$$

$$= \begin{pmatrix} {}^{b}\hat{x}_a^T \\ {}^{b}\hat{y}_a^T \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

So, the rows are orthogonal too!

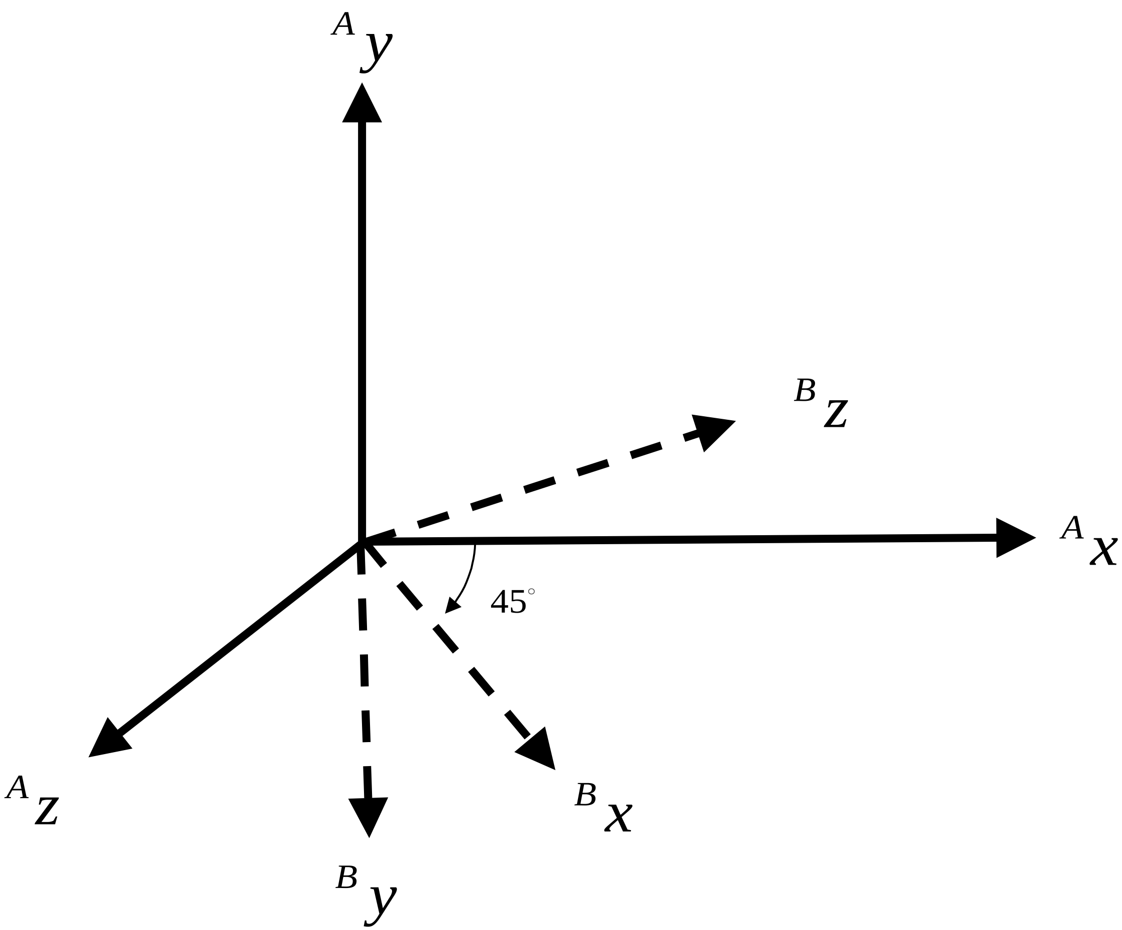
Example 1: rotation matrix



$$^{a}\hat{x}_{b} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \qquad {}^{a}R_{b} = \begin{pmatrix} {}^{a}\hat{x}_{b} & {}^{a}\hat{y}_{b} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$^{a}\hat{y}_{b} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \qquad {}^{b}R_{a} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Example 2: rotation matrix

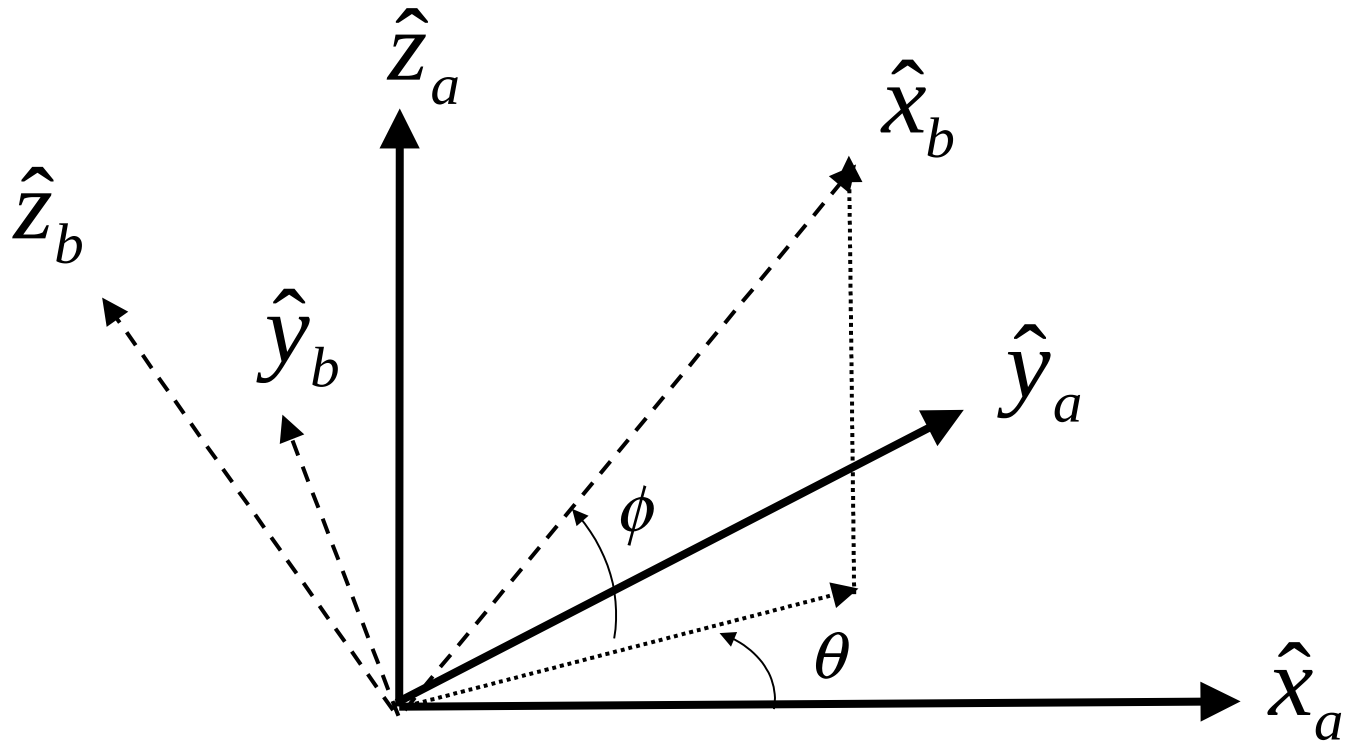


$$^A R_B = \begin{pmatrix} ^A\hat{x}_B & ^A\hat{y}_B & ^A\hat{z}_B \end{pmatrix}$$

$$^A R_B = \left(\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \right)$$

$$^A R_B = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

Example 3: rotation matrix



$$
{}^aR_c = \begin{pmatrix} c_\theta c_\phi & -s_\theta & c_\theta c_{\phi+\frac{\pi}{2}} \\ s_\theta c_\phi & c_\theta & s_\theta c_{\phi+\frac{\pi}{2}} \\ s_\phi & 0 & s_{\phi+\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} c_\theta c_\phi & -s_\theta & -c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & -s_\theta s_\phi \\ s_\phi & 0 & c_\phi \end{pmatrix}
$$

Rotations about x, y, z

$$R_z(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{pmatrix}$$

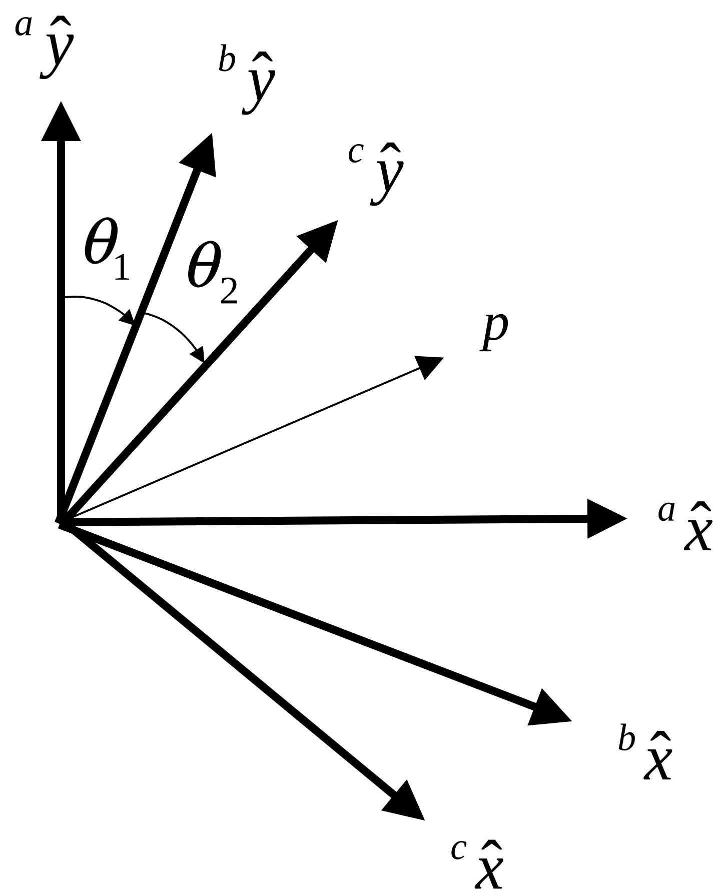
These rotation matrices encode the basis vectors of the after-rotation reference frame in terms of the before-rotation reference frame

Remember those double-angle formulas…

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$

Example 1: composition of rotation matrices

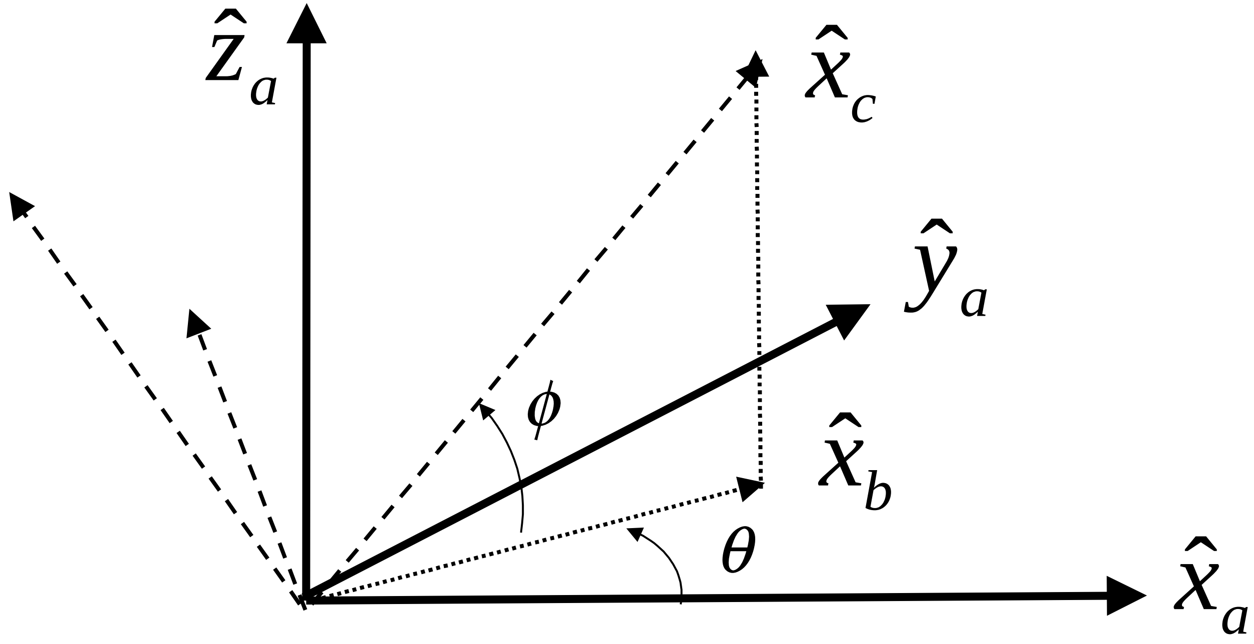


$$ {}^{A}R_{C} = {}^{A}R_{B}\, {}^{B}R_{C} $$

$$ {}^{a}R_{c} = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} = \begin{pmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 \\ s_1c_2 + c_1s_2 & c_1c_2 - s_1s_2 \end{pmatrix} $$

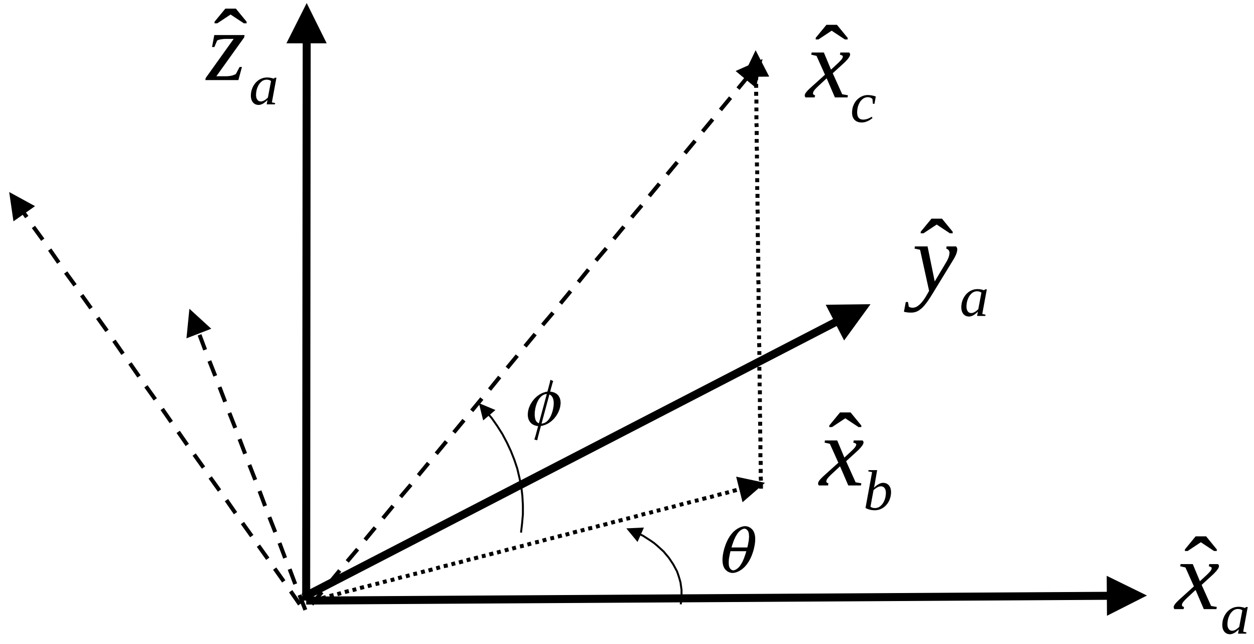
$$ = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} $$

Example 2: composition of rotation matrices



$$
{}^{a}R_{b} = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\qquad
{}^{b}R_{c} = \begin{pmatrix} c_{-\phi} & 0 & s_{-\phi} \\ 0 & 1 & 0 \\ -s_{-\phi} & 0 & c_{-\phi} \end{pmatrix} = \begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix}
$$

Example 2: composition of rotation matrices



$$ {}^aR_c = {}^aR_b\,{}^bR_c = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix} = \begin{pmatrix} c_\theta c_\phi & -s_\theta & -c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & -s_\theta s_\phi \\ s_\phi & 0 & c_\phi \end{pmatrix} $$