Inverse Kinematics

This addresses the obvious question: what joint angles will place my end effector in a desired pose?
Closed form (analytical) solution: a sequence or set of equations that can be solved for the desired joint angles

- Potentially faster than an iterative solution
- A unique solution to all manipulator positions can be determined *a priori*.
  - Can guarantee “safe” joint configurations where the manipulator does not collide with the body.

Iterative (numerical) solution: numerical iteration toward a desired goal position (variation on Newton’s method)

- Easier to think about
- Better suited to incremental displacements and control.
Inverse kinematics

There is no general analytical inverse kinematics solution

- All analytical inverse kinematics solutions are specific to a robot or class of robots.
  - based on geometric intuition about the robot
- I'll give one example – there are many variations.
Inverse kinematics

Consider this 6-joint robot:

- this example is out of the book…

Spherical wrist: the axes of the last three joints intersect in a point.
Inverse kinematics

Problem:
- Given: desired transform, \( T_{\text{eff}} = \begin{bmatrix} R_{\text{eff}} & d_{\text{eff}} \\ 0 & 1 \end{bmatrix} \)
- Find: \( q = (q_1, q_2, q_3, q_4, \ldots, q_n) \)

Note:
- The desired transform (pose) encodes six degrees of freedom (this info can be represented by six numbers)
  - Since we only have six joints at our disposal, there is no manifold of redundant solutions.
- For this manipulator, the problem can be decomposed into a position component (the first three joints) and an orientation component (the last three joints)
- The first three joints tell you what the position of the spherical wrist
Example: Inverse kinematics

Solution:

- First, back out the position of the spherical wrist:

Since it’s a spherical wrist, the last three joints can be thought of as rotating about a point.

- A constant transform exists that goes from the last wrist joint out to the end effector (sometimes this is called the “tool” transform): $^{sw}T_{eff}$

- Back out the position of the wrist: $^{b}T_{sw} = ^{b}T_{eff} \cdot ^{sw}T_{eff}^{-1}$
Example: Inverse kinematics

- Next, solve for the first three joints

First, solve for $q_1$. (look down from above)

$$q_1 = a \tan 2 \left( x_g, y_g \right)$$

or

$$q_1 = a \tan 2 \left( x_g, y_g \right) + \pi$$
Example: Inverse kinematics

Next, solve for \( q_3 \) (look at the manipulator orthogonal to the plane of the first two links)

\[
c^2 = a^2 + b^2 - 2ab \cos(\theta_c)
\]

\[
\cos(\theta_c) = -\frac{r_g^2 + (z_g - h)^2 - l_1^2 - l_2^2}{2l_1l_2} = -D
\]

where \( r_g^2 = x_g^2 + y_g^2 \)

and \( h \) is the height of the first link

\[
\tan(q_3) = \frac{\pm \sqrt{1 - D^2}}{D}
\]
Example: Inverse kinematics

Next, solve for \( q_2 \) (continue to look at the manipulator orthogonal to the plane of the first two links)

\[
\tan(\theta) = \frac{z_g - h}{\sqrt{x_g^2 + y_g^2}}
\]

\[
\tan(\alpha) = \frac{l_2 s_3}{l_1 + l_2 c_3}
\]

\[
q_2 = \theta \pm \alpha
\]
Finally, the last three joints completely specify the orientation of the end effector.

- Note that the last three joints look just like ZYZ Euler angles
  - Determination of the joint angles is easy – just calculate the ZYZ Euler angles corresponding to the desired orientation.
Remember: ZYZ Euler Angles

\[
R_{z_y_z}(\phi, \theta, \psi) = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
R_{z_y_z}(\phi, \theta, \psi) = \begin{pmatrix}
c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\
s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\
-s_\theta c_\psi & s_\theta s_\psi & c_\theta
\end{pmatrix}
\]

\[
\theta = \pm a \tan 2\left(\sqrt{1-r_{33}^{2}}, r_{33}\right)
\]

\[
\phi = a \tan 2\left(r_{23}, r_{13}\right) + k\pi
\]

\[
\psi = a \tan 2\left(r_{32}, r_{31}\right)
\]
Inverse kinematics for a humanoid arm

You can do similar types of things for a humanoid (7-DOF) arm.

• Since this is a redundant arm, there are a manifold of solutions…

General strategy:

1. Solve for elbow angle
2. Solve for a set of shoulder angles that places the wrist in the right position (note that you have to choose an elbow orbit angle)
3. Solve for the wrist angles