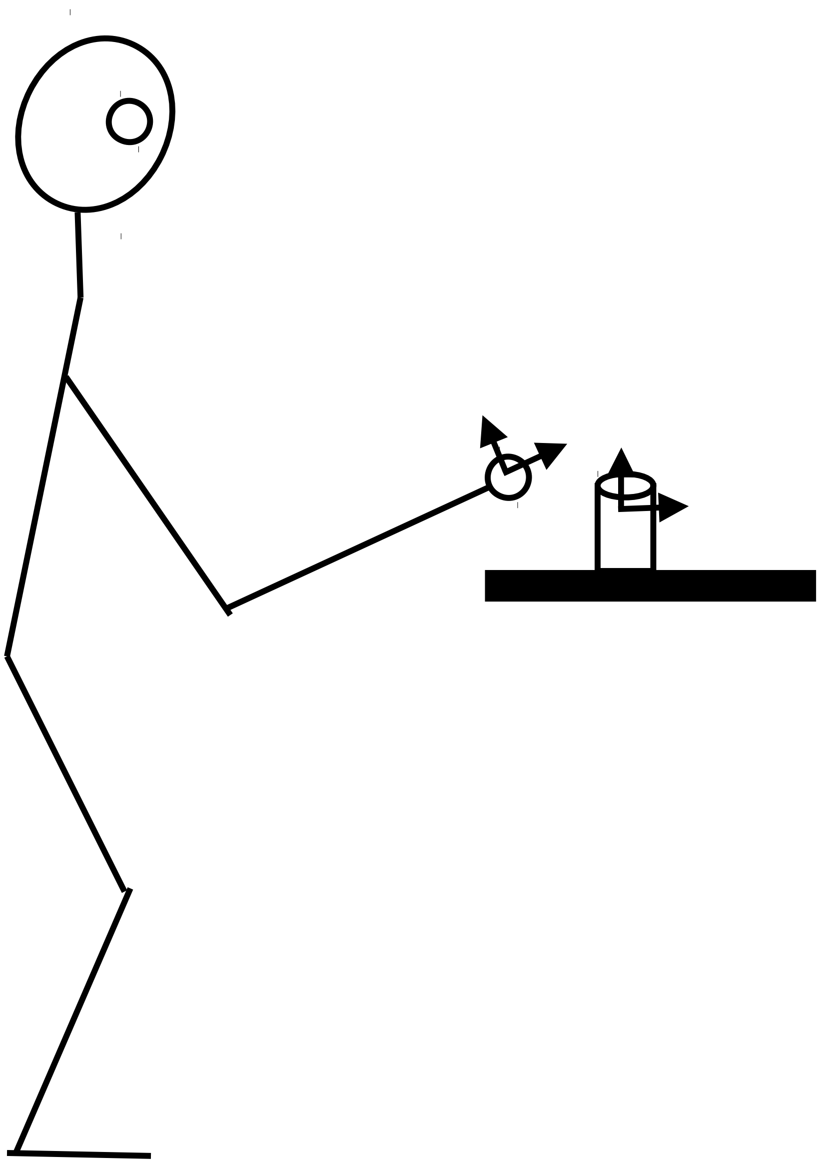
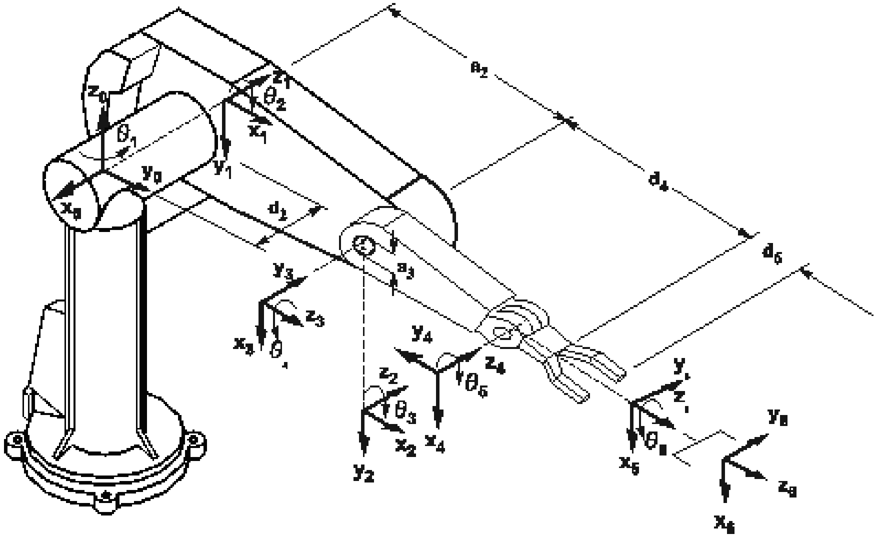
Vectors, Matrices, Rotations

Why are we studying this?

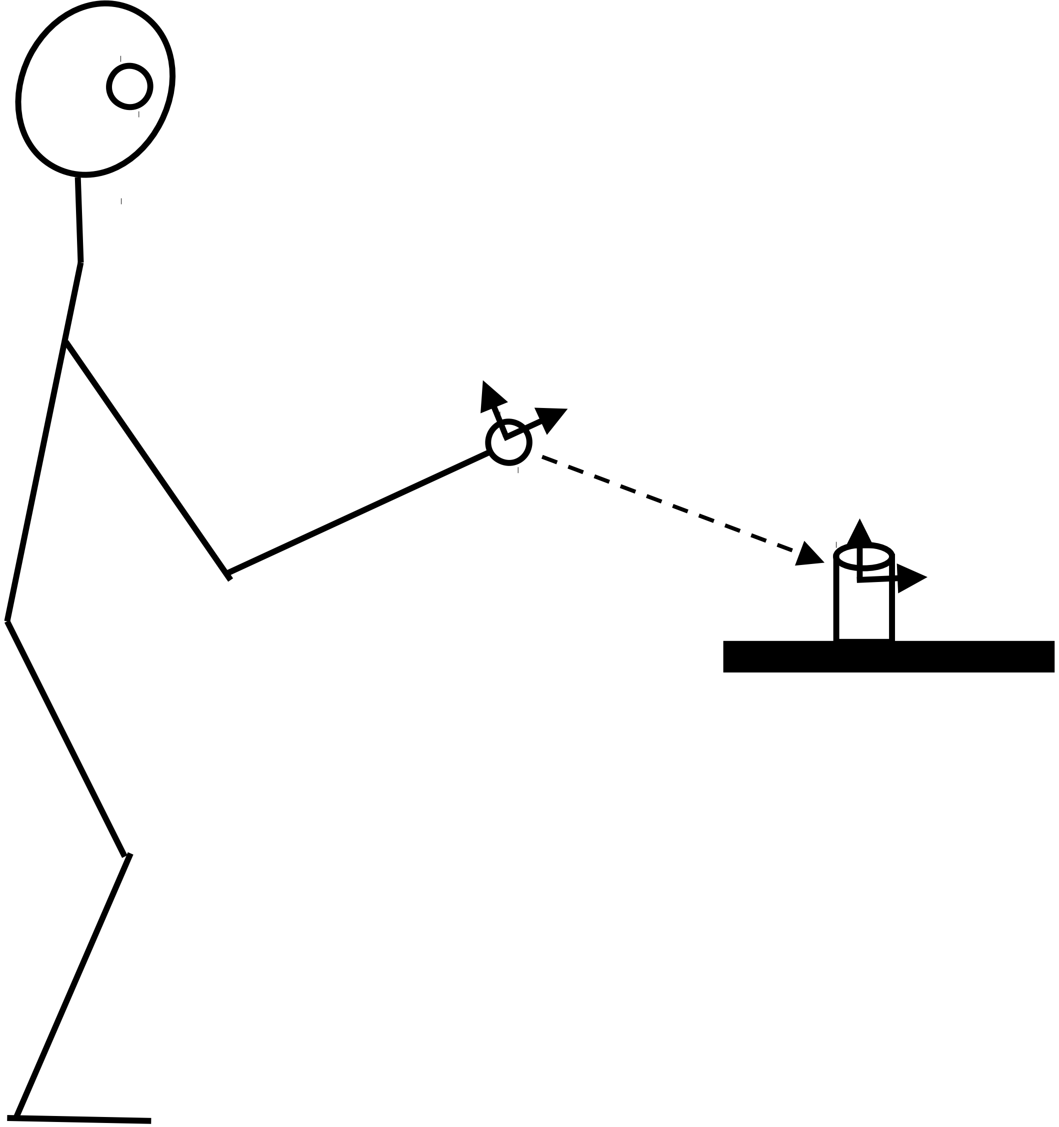


You want to put your hand on the cup…

- Suppose your eyes tell you where the mug is and its orientation in the robot *base frame* (big assumption)
- In order to put your hand on the object, you want to align the coordinate frame of your hand w/ that of the object
- This kind of problem makes representation of pose important...



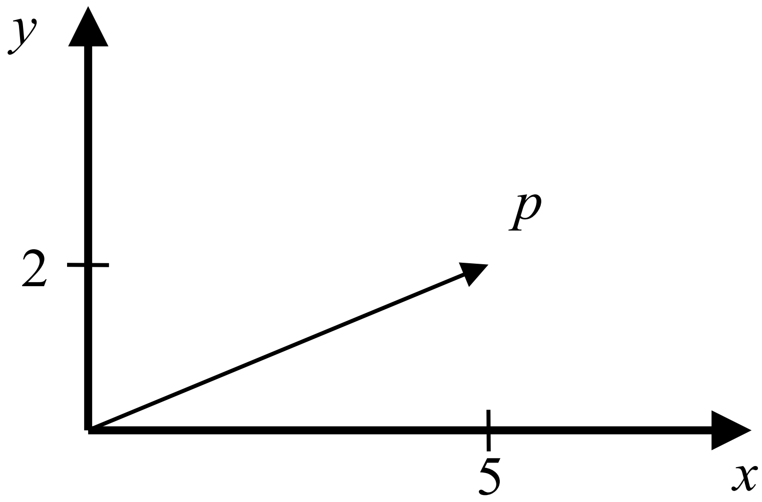
Representing Position: Vectors



Representing Position: vectors

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

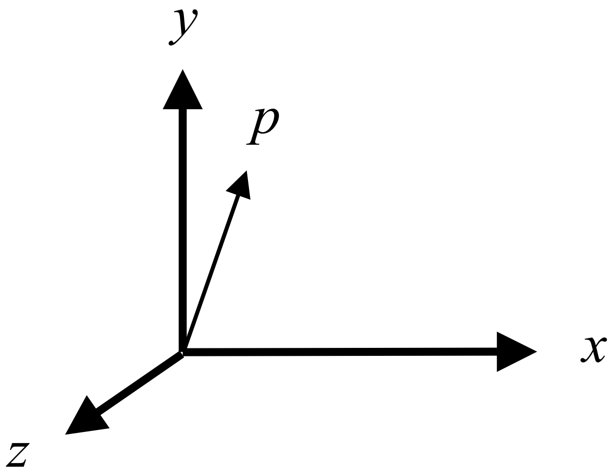
("column" vector)



$$p = \begin{bmatrix} 2 & 5 \end{bmatrix}$$

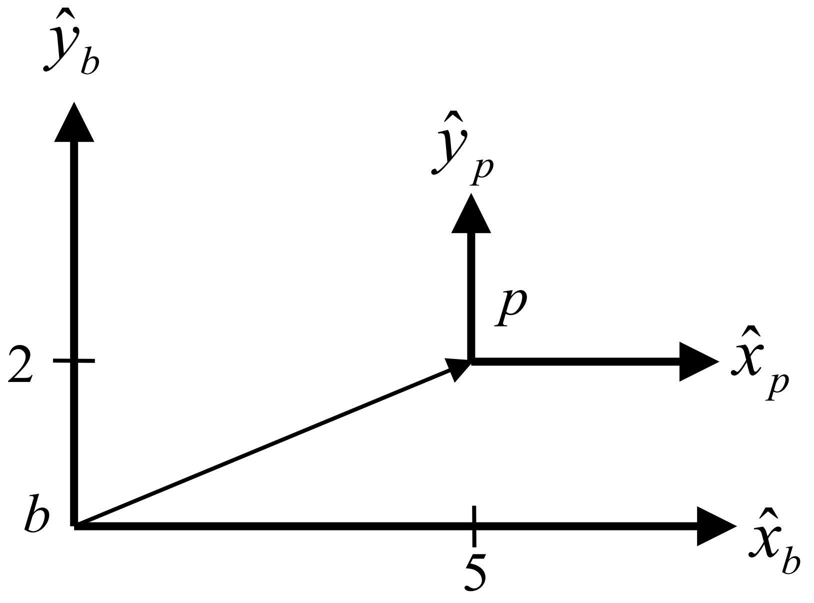
("row" vector)

$$p = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$



Representing Position: vectors

- Vectors are a way to transform between two different reference frames w/ the same orientation
- The prefix superscript denotes the reference frame in which the vector should be understood



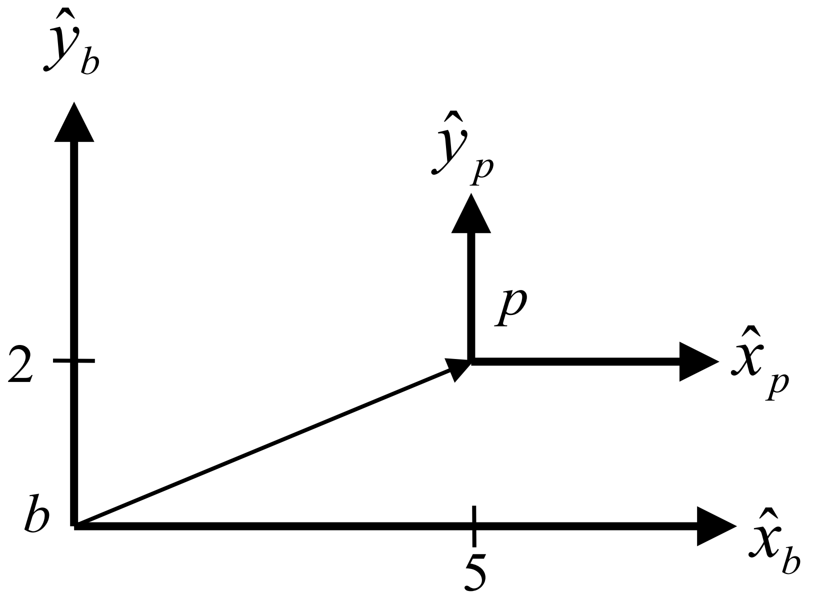
$$ {}^{b}p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad {}^{p}p = \begin{bmatrix} 0 \\ 0 \end{bmatrix} $$

Same point, two different reference frames

Representing Position: vectors

- Note that I am denoting the axes as *orthogonal* unit basis vectors

This means “perpendicular”



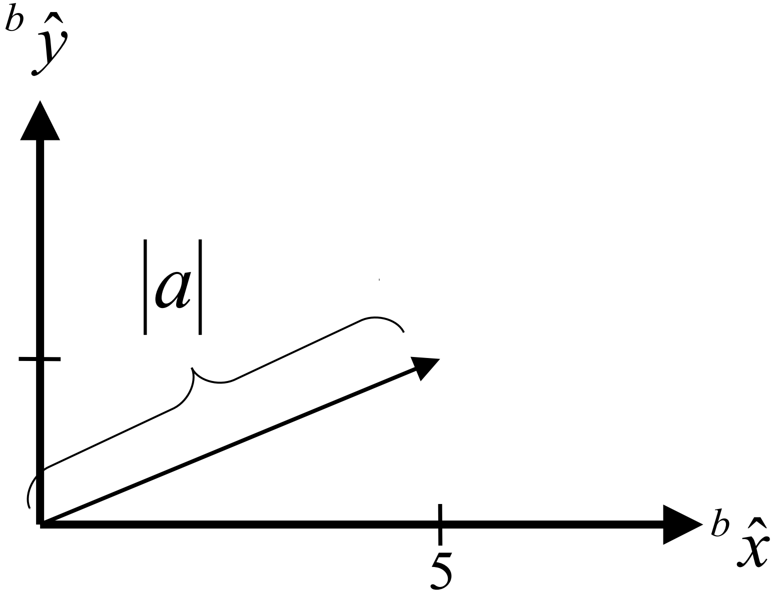
$\hat{x}_b$ ← A vector of length one pointing in the direction of the base frame *x* axis

$\hat{y}_b$ ← *y* axis

$\hat{y}_p$ ← *p* frame y axis

What is this unit vector you speak of?

These are the elements of *a*: $$a = \begin{bmatrix} a_x \\ a_y \end{bmatrix}$$



Vector length/magnitude: $$|a| = \sqrt{a_x^2 + a_y^2}$$

Definition of unit vector: $$|\hat{a}| = 1$$

You can turn *a* into a unit vector of the same direction this way: $$\hat{a} = \frac{a}{\sqrt{a_x^2 + a_y^2}}$$

And what does orthogonal mean?

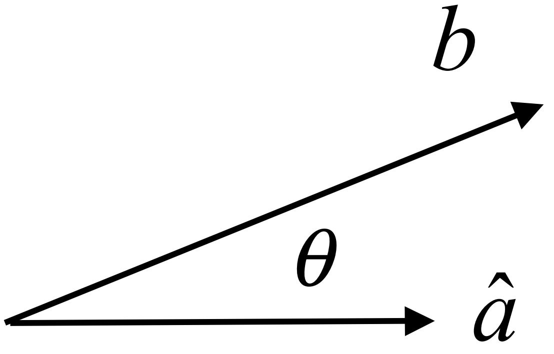
First, define the dot product: $a \cdot b = a_x b_x + a_y b_y$

$$= |a||b|\cos(\theta)$$

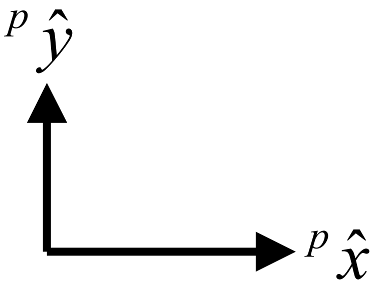
$a \cdot b = 0$ when: $a = 0$

or, $b = 0$

or, $\cos(\theta) = 0$



Unit vectors are orthogonal *iff* (if and only if) the dot product is zero:



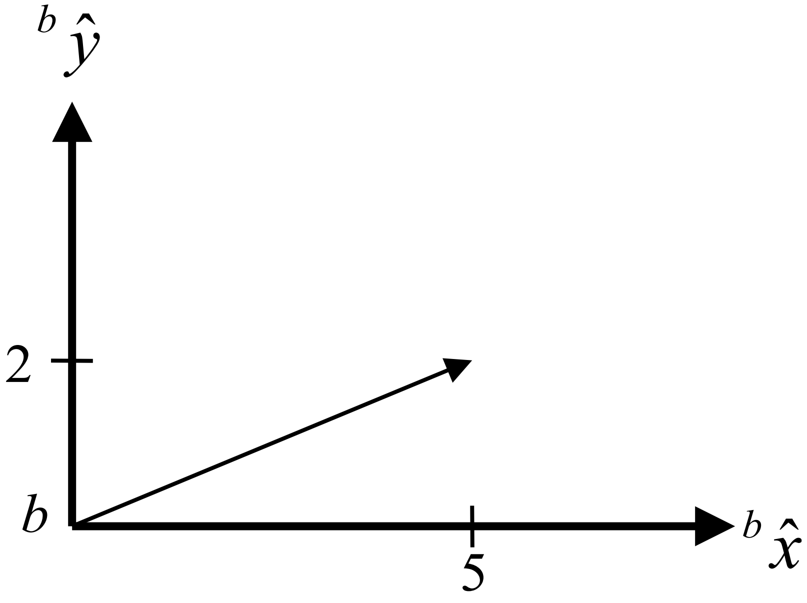
${}^{p}\hat{x}$ is orthogonal to ${}^{p}\hat{y}$ if ${}^{p}\hat{x} \cdot {}^{p}\hat{y} = 0$

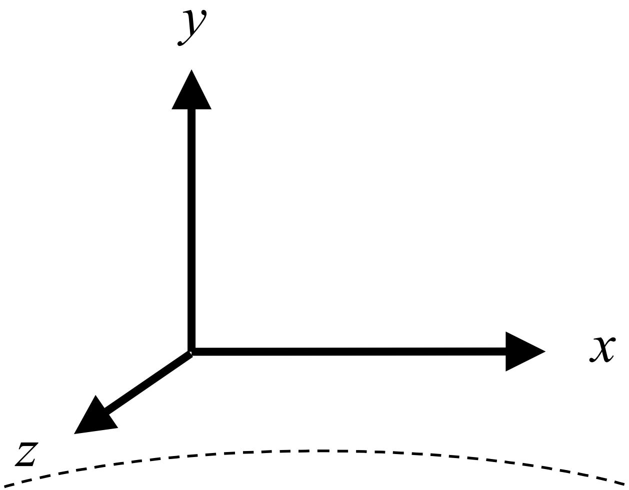
f

A couple of other random things

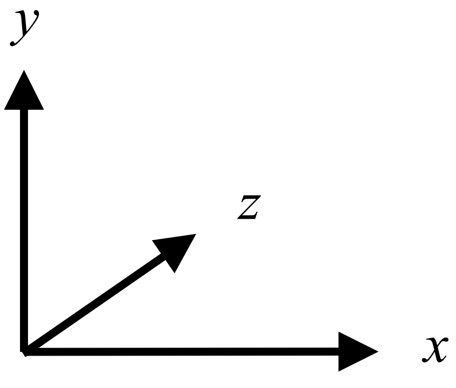
$$p_b = 5\hat{x}_b + 2\hat{y}_b$$

Vectors are elements of R^n





right-handed
coordinate frame

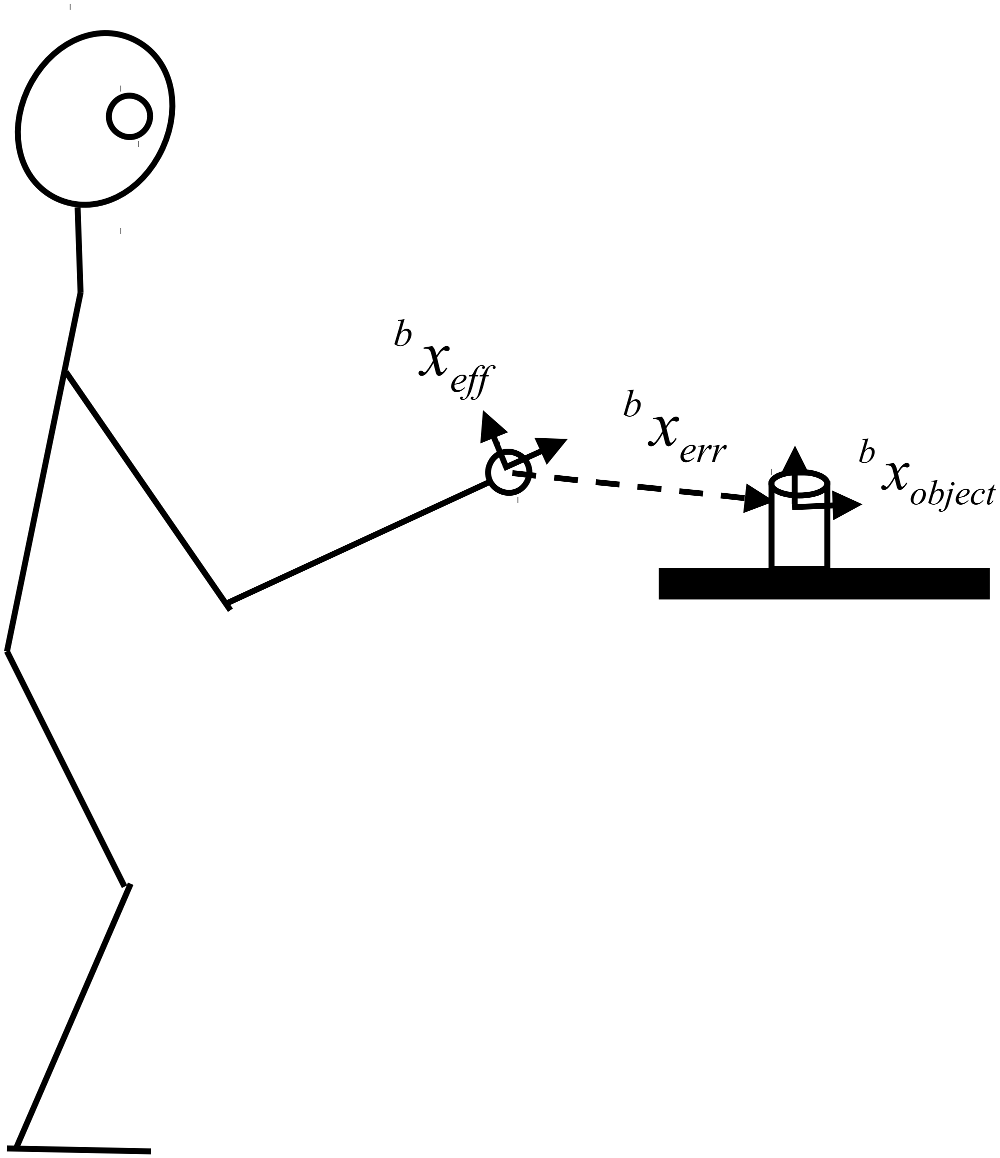


left-handed
coordinate frame

The importance of differencing two vectors

$$ {}^{b}x_{object} - {}^{b}x_{eff} = {}^{b}x_{err} $$

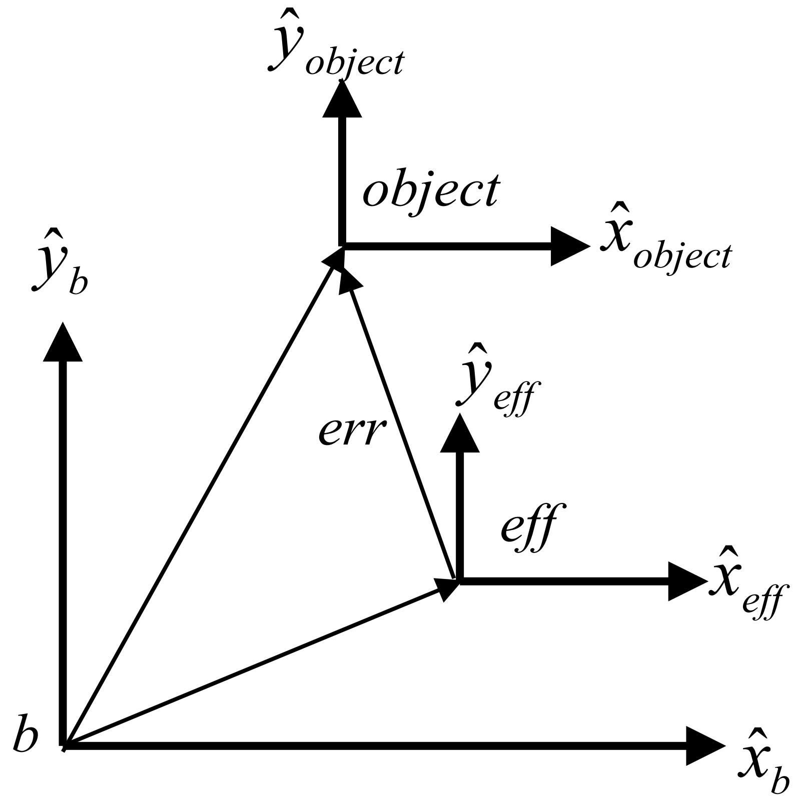
The *eff* needs to make a Cartesian displacement of this much to reach the object



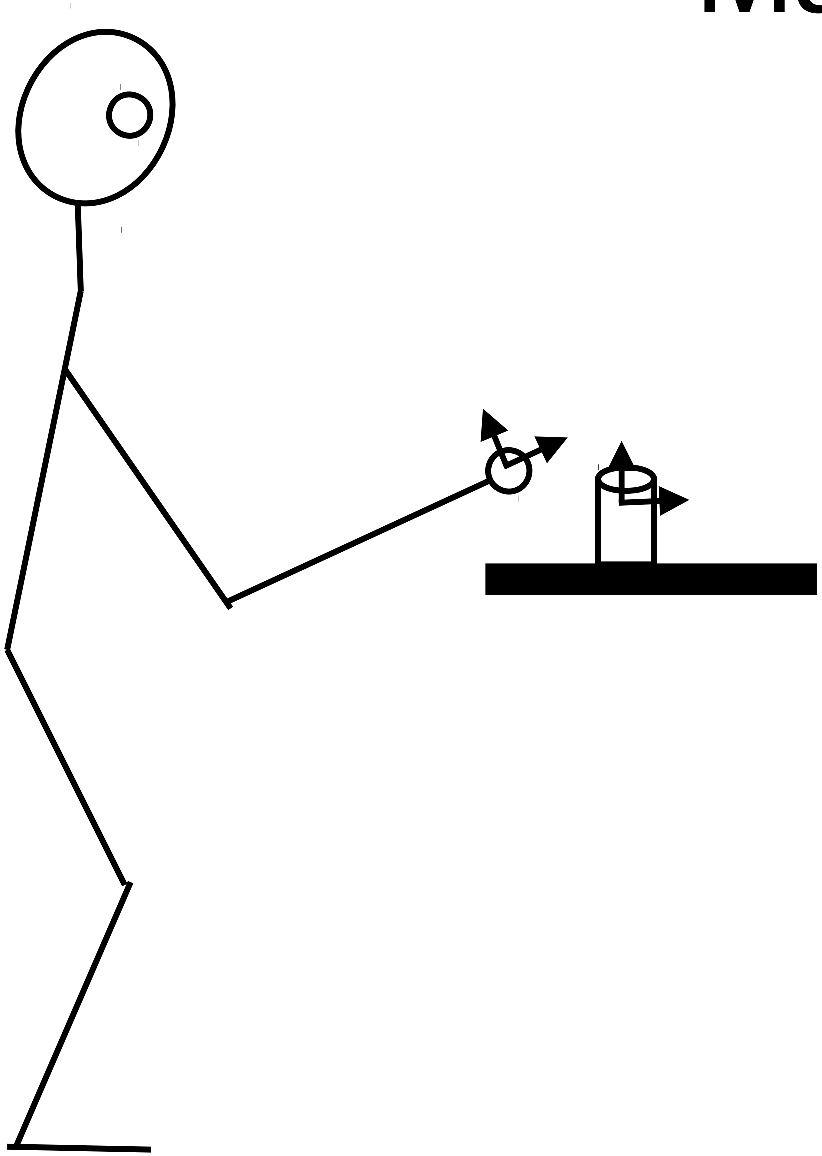
The importance of differencing two vectors

$$^{b}x_{object} - {}^{b}x_{eff} = {}^{b}x_{err}$$

The *eff* needs to make a Cartesian displacement of this much to reach the object

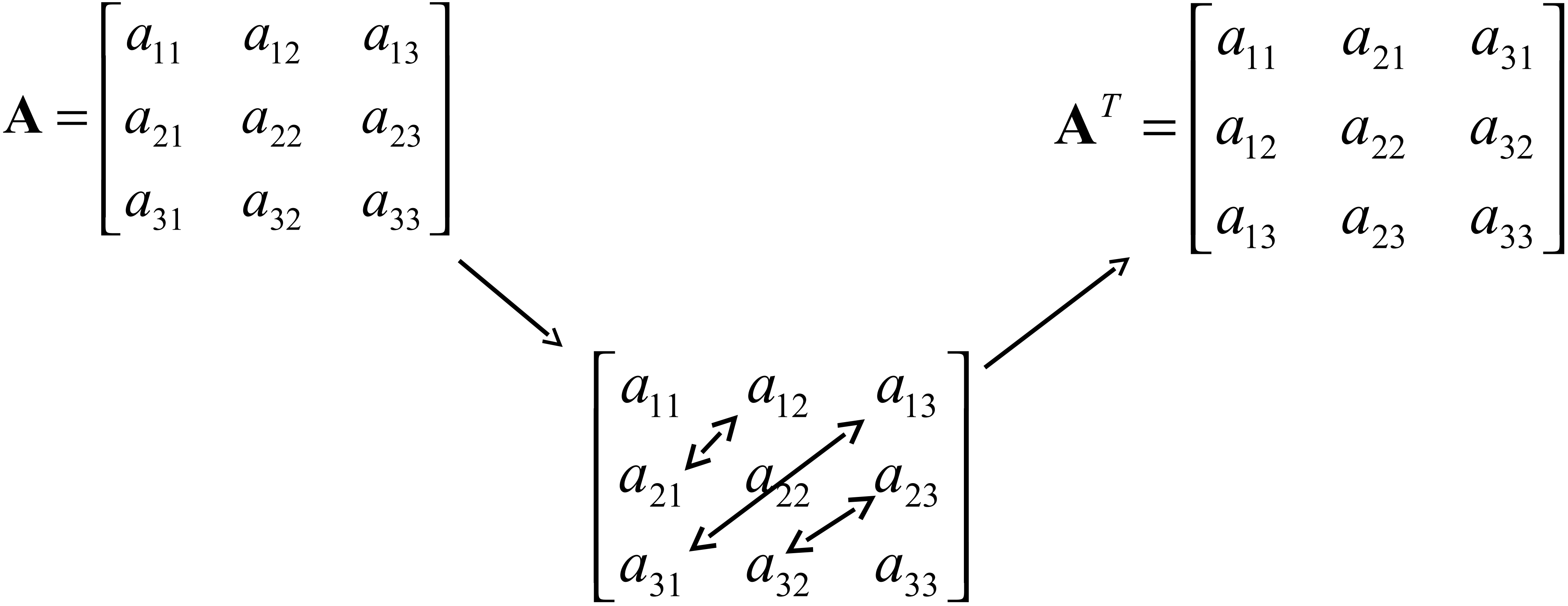


Representing Orientation: Rotation Matrices



- The reference frame of the hand and the object have different orientations
- We want to represent and difference orientations just like we did for positions…

Before we go there – review of matrix transpose



$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \longrightarrow p^T = \begin{bmatrix} 5 & 2 \end{bmatrix}$$

Important property: $\mathbf{A}^T \mathbf{B}^T = \left(\mathbf{BA}\right)^T$

and matrix multiplication…

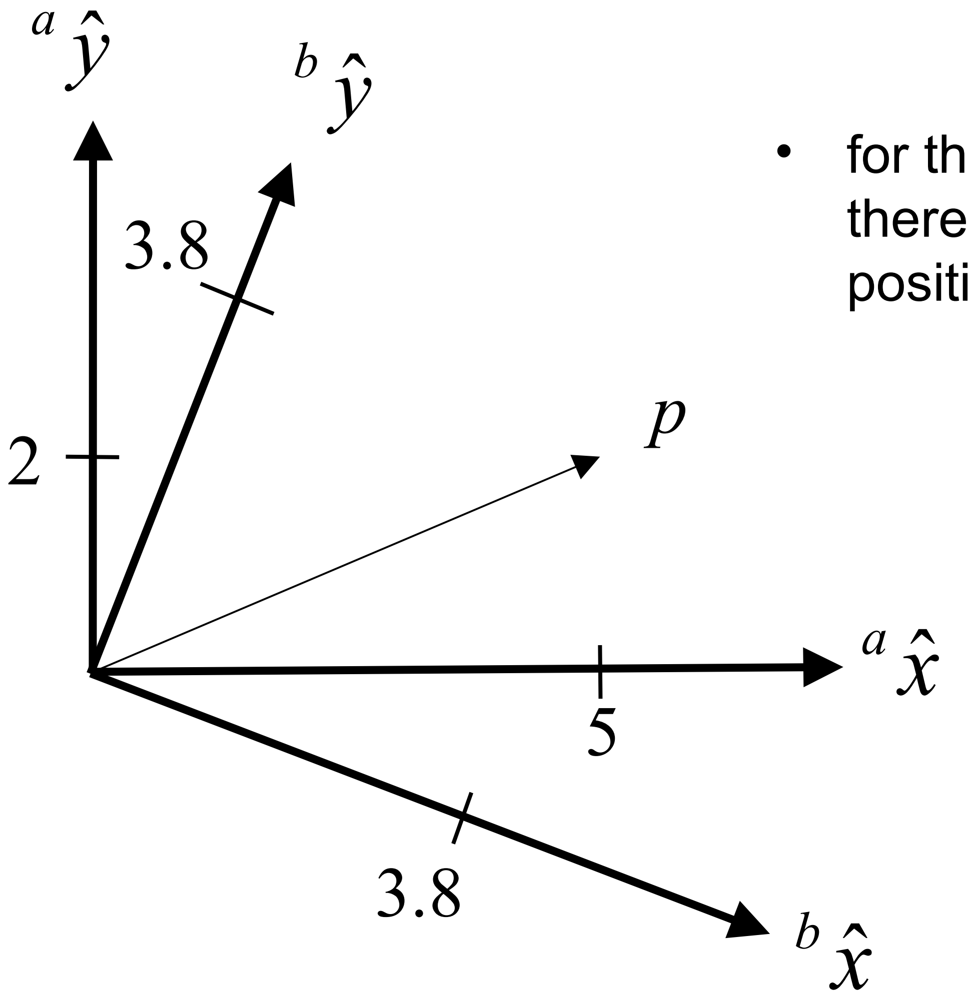
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Can represent dot product as a matrix multiply:

$$a \cdot b = a_x b_x + a_y b_y = \begin{bmatrix} a_x & a_y \end{bmatrix} \begin{bmatrix} b_x \\ b_y \end{bmatrix} = a^T b$$

Same point - different reference frames



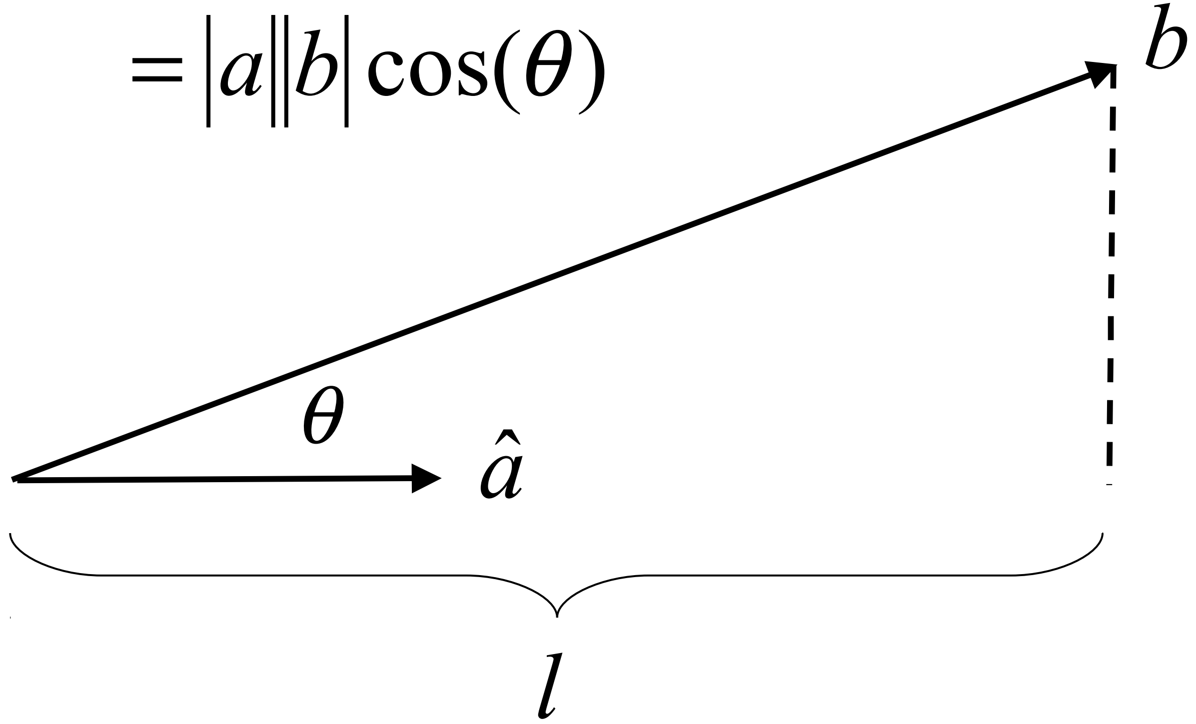
- for the moment, assume that there is no difference in position…

$${}^{a}p = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad {}^{b}p = \begin{bmatrix} 3.8 \\ 3.8 \end{bmatrix}$$

Another important use of the dot product: projection

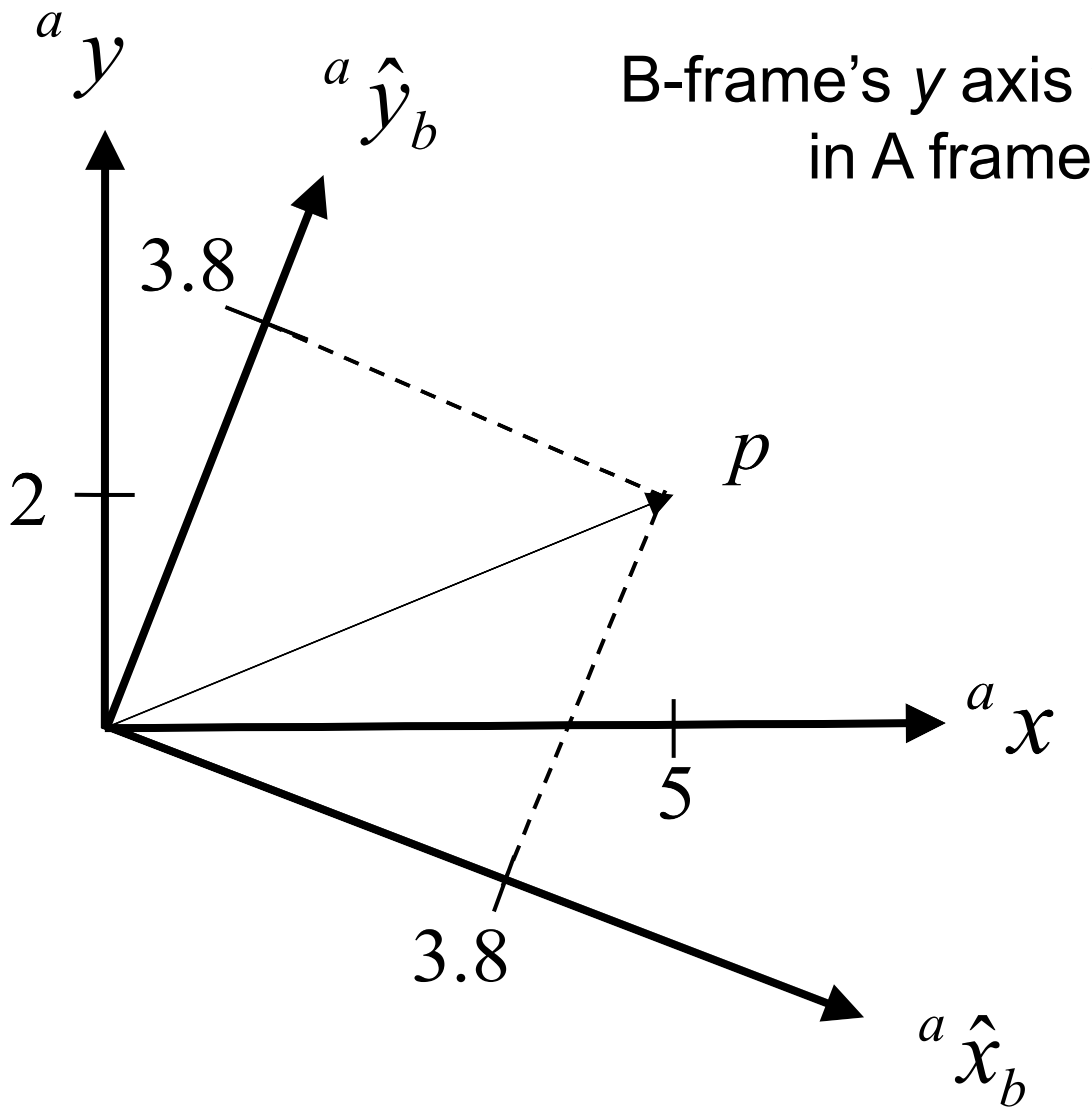
$$a \cdot b = a_x b_x + a_y b_y$$

$$= |a||b|\cos(\theta)$$



$$l = \hat{a} \cdot b = |\hat{a}||b|\cos(\theta) = |b|\cos(\theta)$$

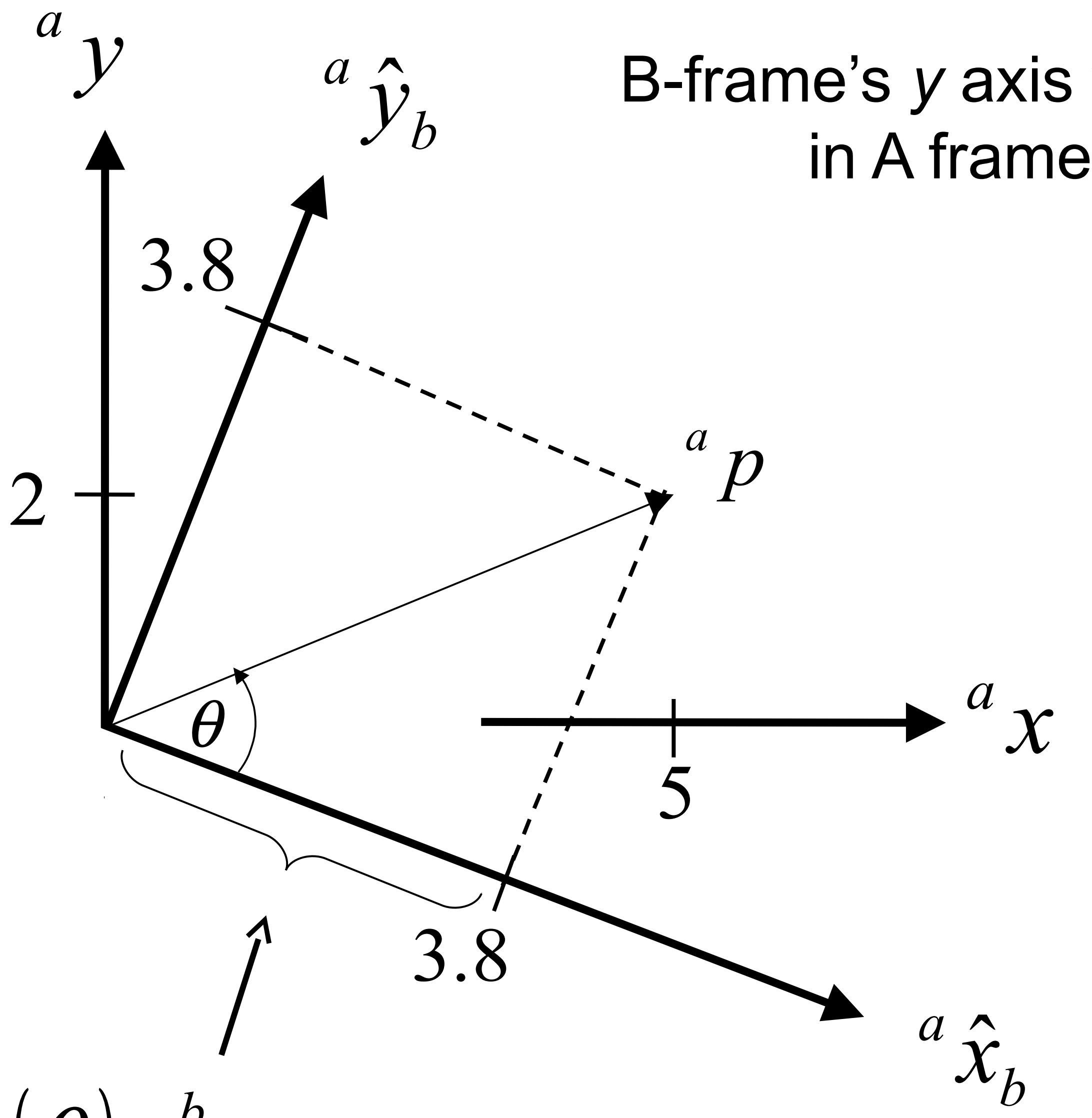
Same point - different reference frames



B-frame's *y* axis written
in A frame

B-frame's *x* axis written
in A frame

Same point - different reference frames

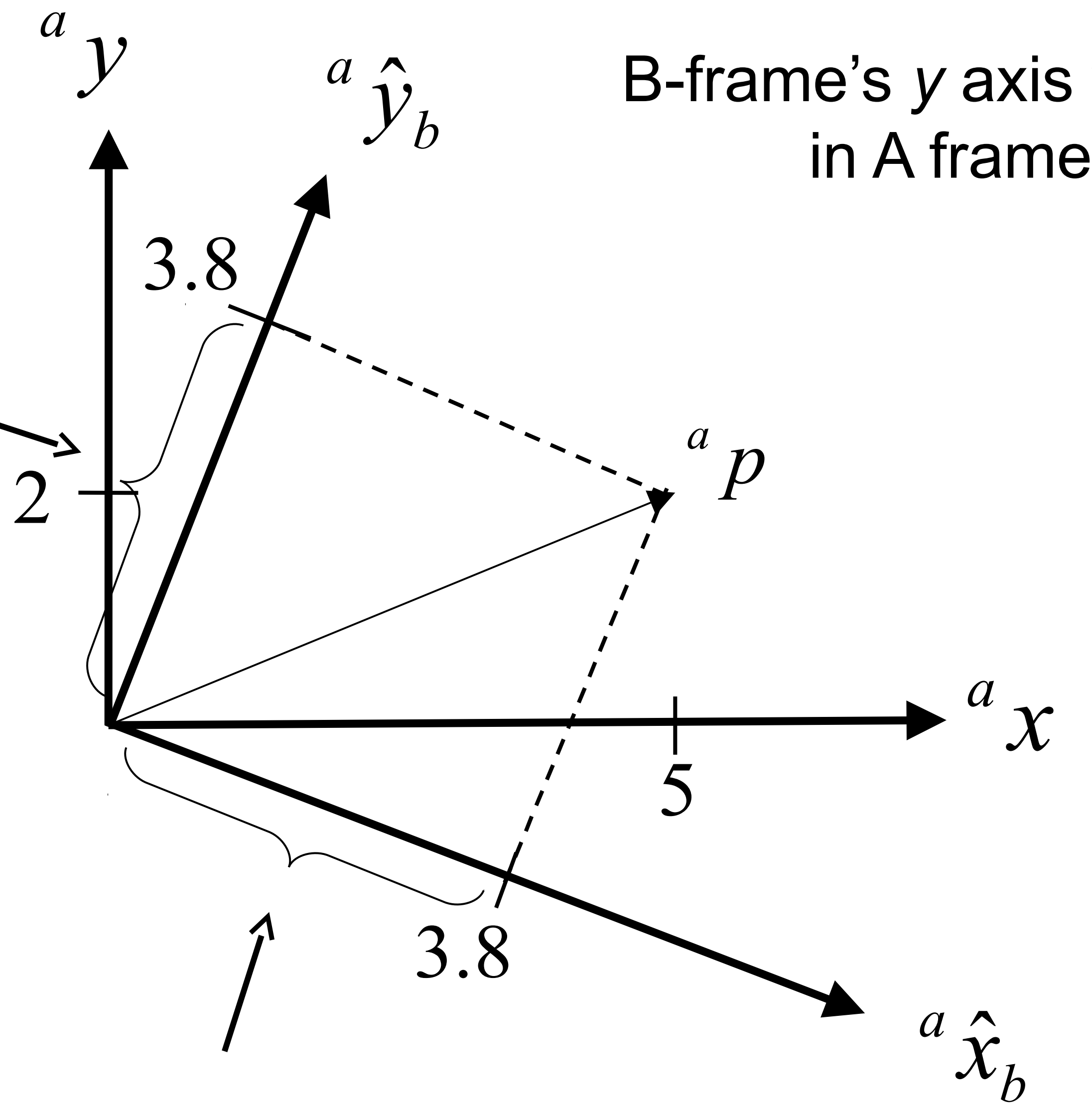


B-frame's *y* axis written
in A frame

$${}^{a}\hat{x}_b \cdot {}^{a}p = {}^{a}p\cos(\theta) = {}^{b}p_x$$

B-frame's *x* axis written
in A frame

Same point - different reference frames



B-frame’s *y* axis written in A frame

$${}^b p_y = {}^a\hat{y}_b \cdot {}^a p$$

$${}^b p_x = {}^a\hat{x}_b \cdot {}^a p$$

B-frame’s *x* axis written in A frame

Same point - different reference frames

$$ {}^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B \cdot {}^{A}p \\ {}^{A}\hat{y}_B \cdot {}^{A}p \end{pmatrix} = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T}\,{}^{A}p \\ {}^{A}\hat{y}_B{}^{T}\,{}^{A}p \end{pmatrix} = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T} \\ {}^{A}\hat{y}_B{}^{T} \end{pmatrix} {}^{A}p $$

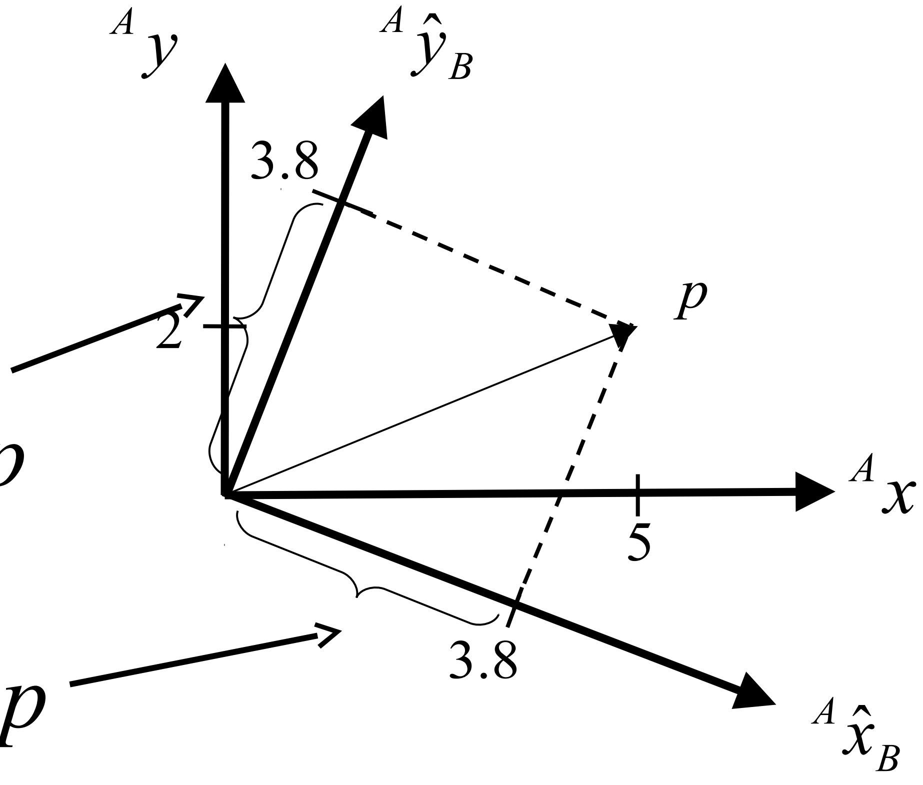
$$ {}^{B}p = \begin{pmatrix} {}^{A}\hat{x}_B{}^{T} \\ {}^{A}\hat{y}_B{}^{T} \end{pmatrix} {}^{A}p $$

$$ {}^{B}p = {}^{A}R_B{}^{T}\,{}^{A}p $$

Rotation matrix

$$ {}^{B}p_y = {}^{A}\hat{y}_B \cdot {}^{A}p $$

$$ {}^{B}p_x = {}^{A}\hat{x}_B \cdot {}^{A}p $$



The rotation matrix

From last page: $${}^{B}p = \begin{pmatrix} {}^{A}\hat{x}_{B}{}^{T} \\ {}^{A}\hat{y}_{B}{}^{T} \end{pmatrix} {}^{A}p \longrightarrow {}^{B}p = {}^{A}R_{B}{}^{T}\,{}^{A}p$$

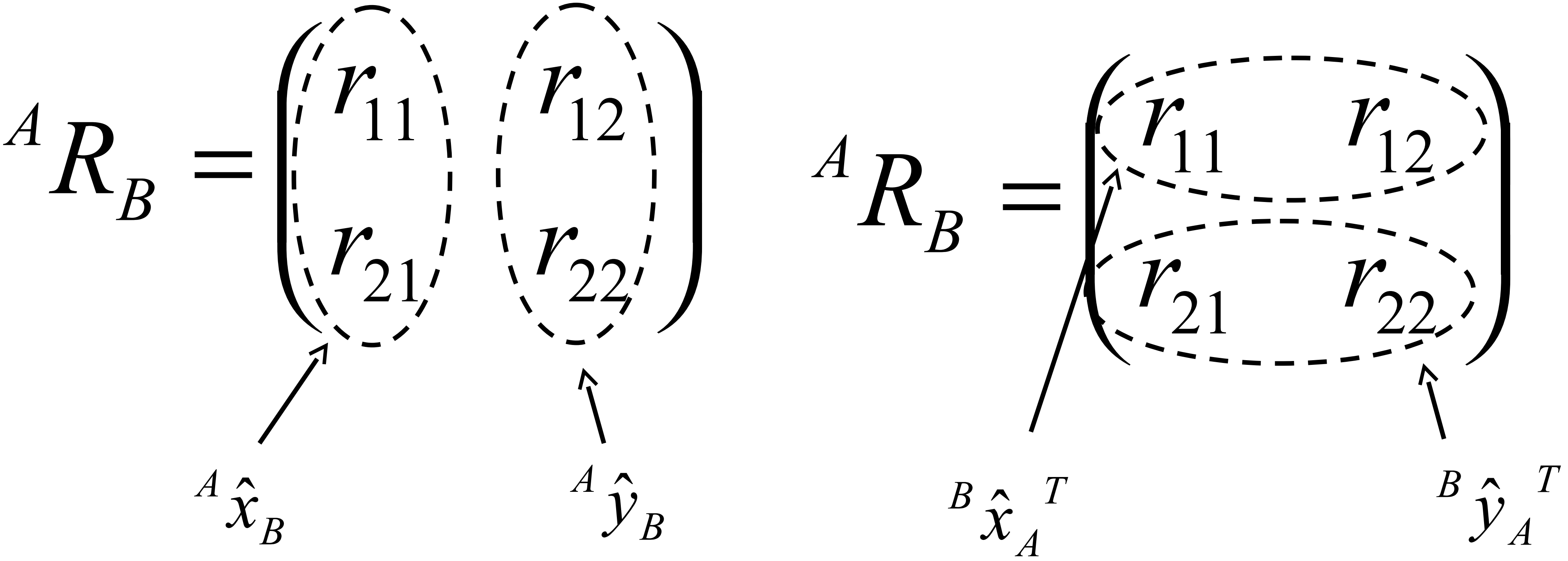
By the same reasoning: $${}^{A}p = \begin{pmatrix} {}^{B}\hat{x}_{A}{}^{T} \\ {}^{B}\hat{y}_{A}{}^{T} \end{pmatrix} {}^{B}p \longrightarrow {}^{A}p = {}^{B}R_{A}{}^{T}\,{}^{B}p$$

The rotation matrix

$${}^{A}R_{B} = \begin{pmatrix} {}^{A}\hat{x}_{B} & {}^{A}\hat{y}_{B} \end{pmatrix}$$

and

$${}^{A}R_{B} = {}^{B}R_{A}{}^{T} = \begin{pmatrix} {}^{B}\hat{x}_{A}{}^{T} \\ {}^{B}\hat{y}_{A}{}^{T} \end{pmatrix}$$



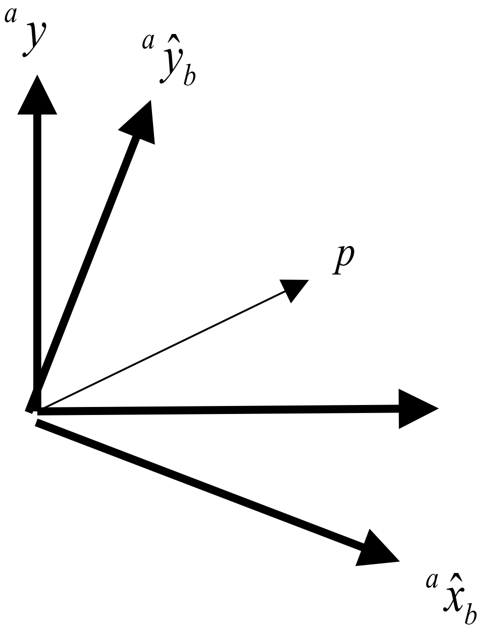
$${}^{A}R_{B} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \qquad {}^{A}R_{B} = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$$

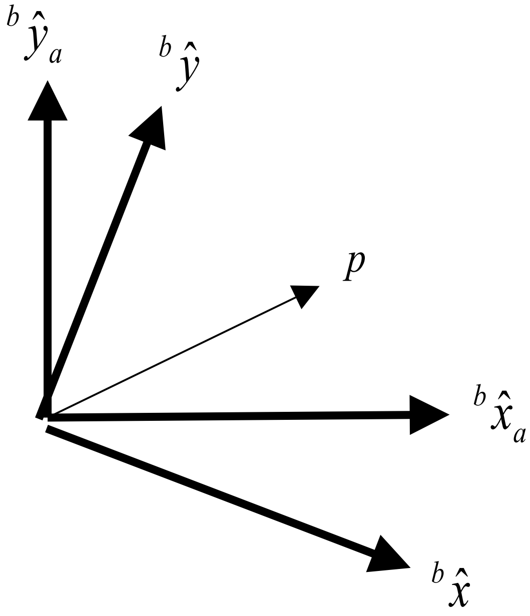
$${}^{A}\hat{x}_{B} \qquad {}^{A}\hat{y}_{B} \qquad {}^{B}\hat{x}_{A}{}^{T} \qquad {}^{B}\hat{y}_{A}{}^{T}$$

The rotation matrix can be understood as:

1. Columns of vectors of *B* in *A* reference frame, OR
2. Rows of column vectors *A* in *B* reference frame

The rotation matrix

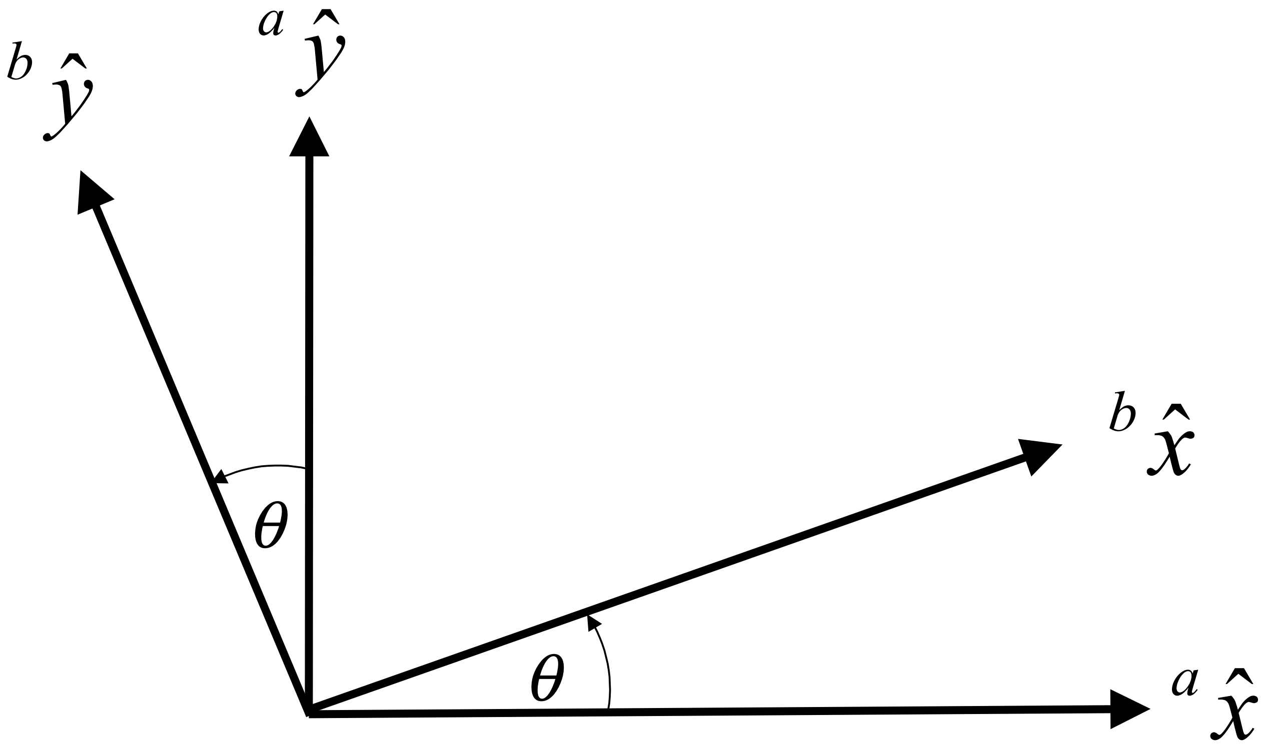




$$ {}^A R_B = \left({}^A\hat{x}_B \quad {}^A\hat{y}_B \right) $$

$$ {}^A R_B = \begin{pmatrix} {}^B\hat{x}_A{}^T \\ {}^B\hat{y}_A{}^T \end{pmatrix} $$

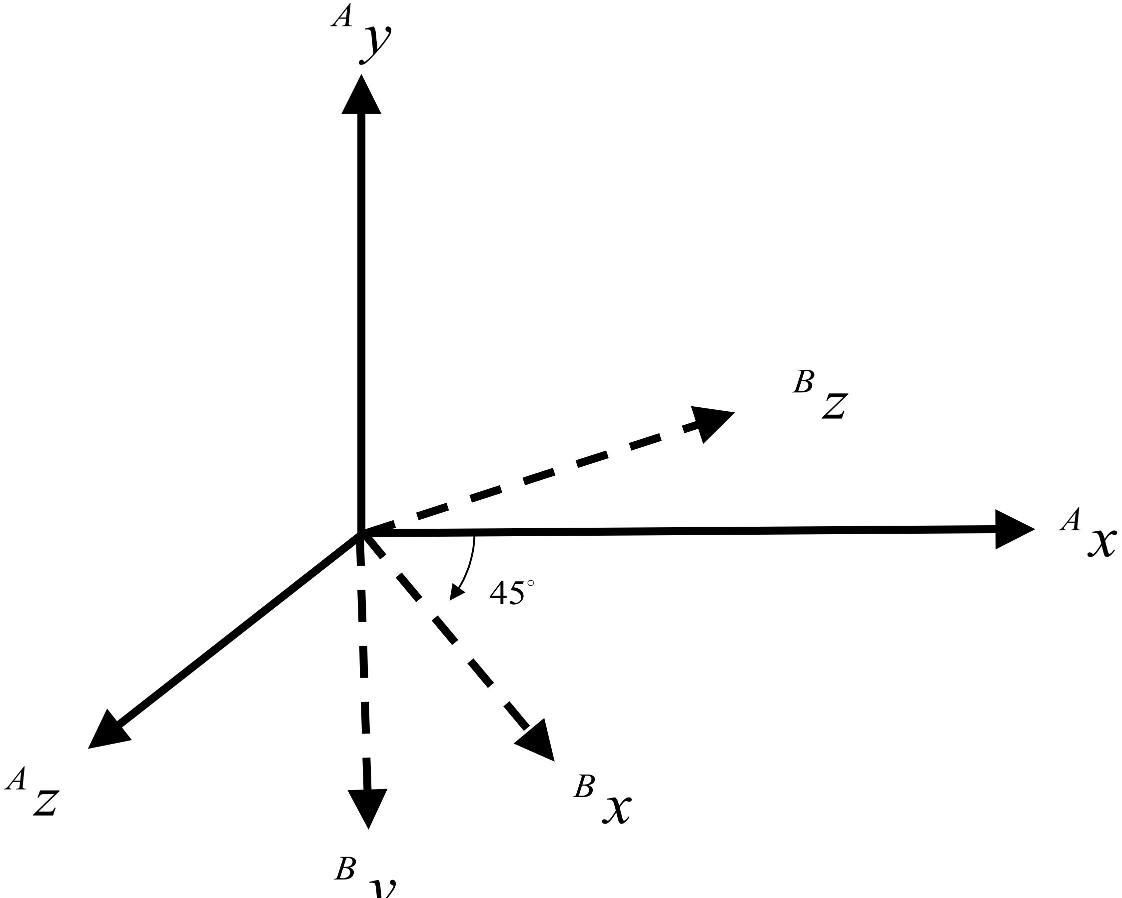
Example 1: rotation matrix



$${}^{a}\hat{x}_b = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \qquad {}^{a}R_b = \begin{pmatrix} {}^{a}\hat{x}_b & {}^{a}\hat{y}_b \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$${}^{a}\hat{y}_b = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \qquad {}^{b}R_a = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Example 2: rotation matrix

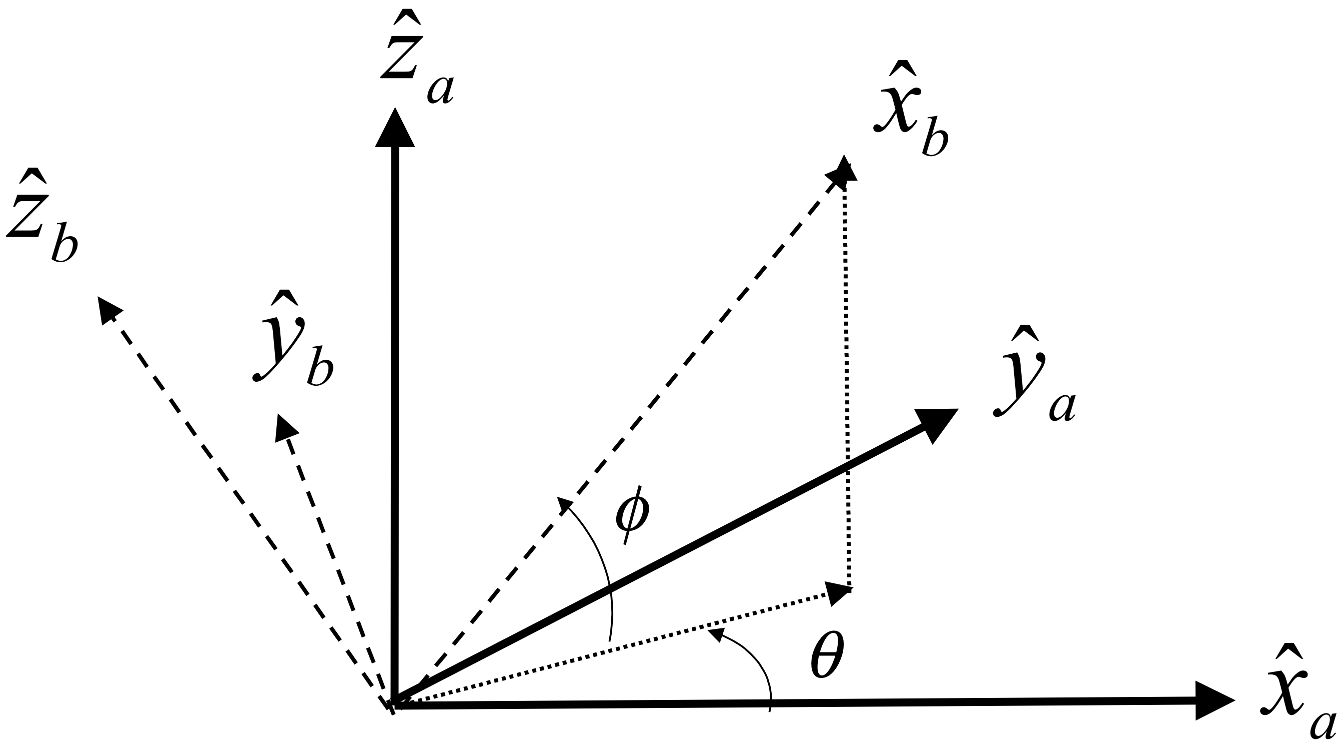


$${}^{A}R_{B} = \left({}^{A}\hat{x}_{B} \quad {}^{A}\hat{y}_{B} \quad {}^{A}\hat{z}_{B} \right)$$

$${}^{A}R_{B} = \left(\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \right)$$

$${}^{A}R_{B} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$$

Example 3: rotation matrix



$$ {}^aR_c = \begin{pmatrix} c_\theta c_\phi & -s_\theta & c_\theta c_{\phi+\frac{\pi}{2}} \\ s_\theta c_\phi & c_\theta & s_\theta c_{\phi+\frac{\pi}{2}} \\ s_\phi & 0 & s_{\phi+\frac{\pi}{2}} \end{pmatrix} = \begin{pmatrix} c_\theta c_\phi & -s_\theta & -c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & -s_\theta s_\phi \\ s_\phi & 0 & c_\phi \end{pmatrix} $$

Rotations about x, y, z

$$R_z(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{pmatrix}$$

$$R_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\gamma) & -\sin(\gamma) \\ 0 & \sin(\gamma) & \cos(\gamma) \end{pmatrix}$$

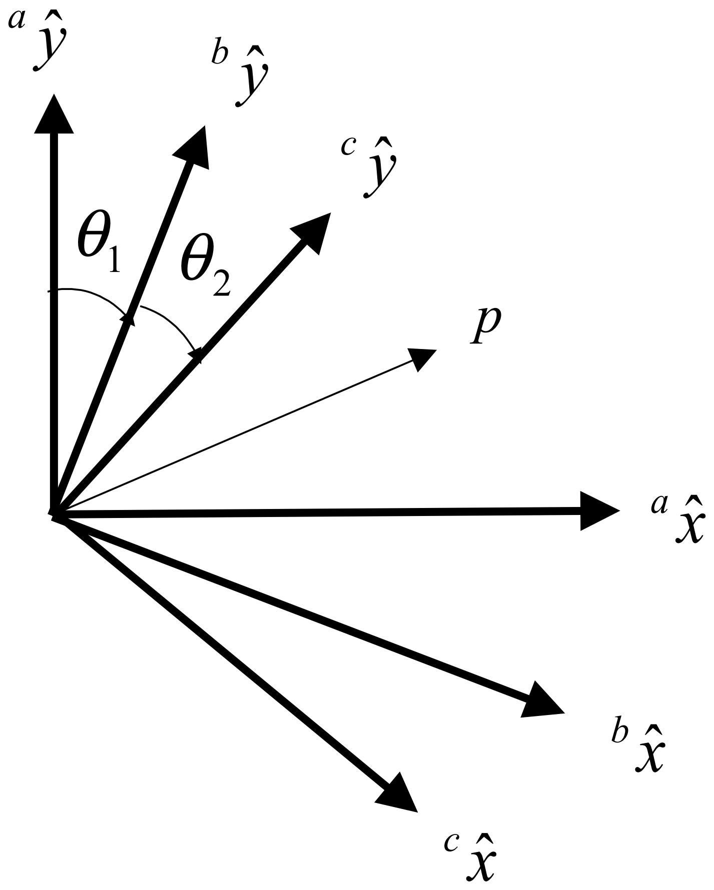
These rotation matrices encode the basis vectors of the after-rotation reference frame in terms of the before-rotation reference frame

Remember those double-angle formulas…

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \mp \sin(\theta)\sin(\phi)$$

Example 1: composition of rotation matrices

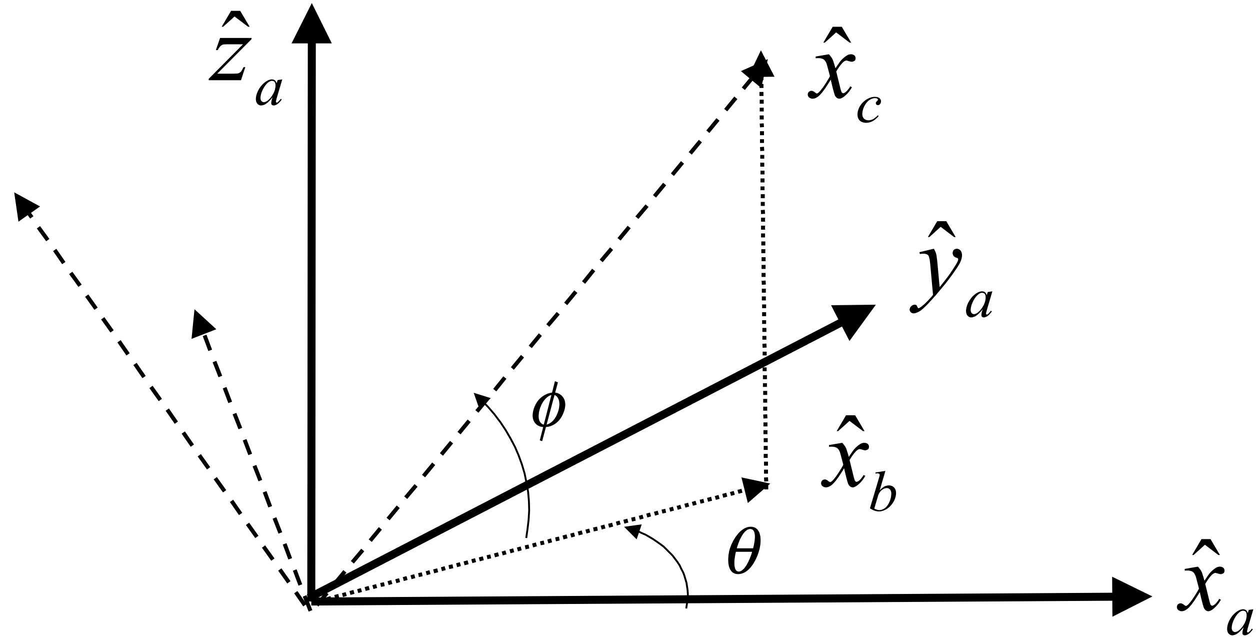


$$ {}^{A}R_{C} = {}^{A}R_{B}\, {}^{B}R_{C} $$

$$ {}^{a}R_{c} = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix} = \begin{pmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 \\ s_1c_2 + c_1s_2 & c_1c_2 - s_1s_2 \end{pmatrix} $$

$$ = \begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} $$

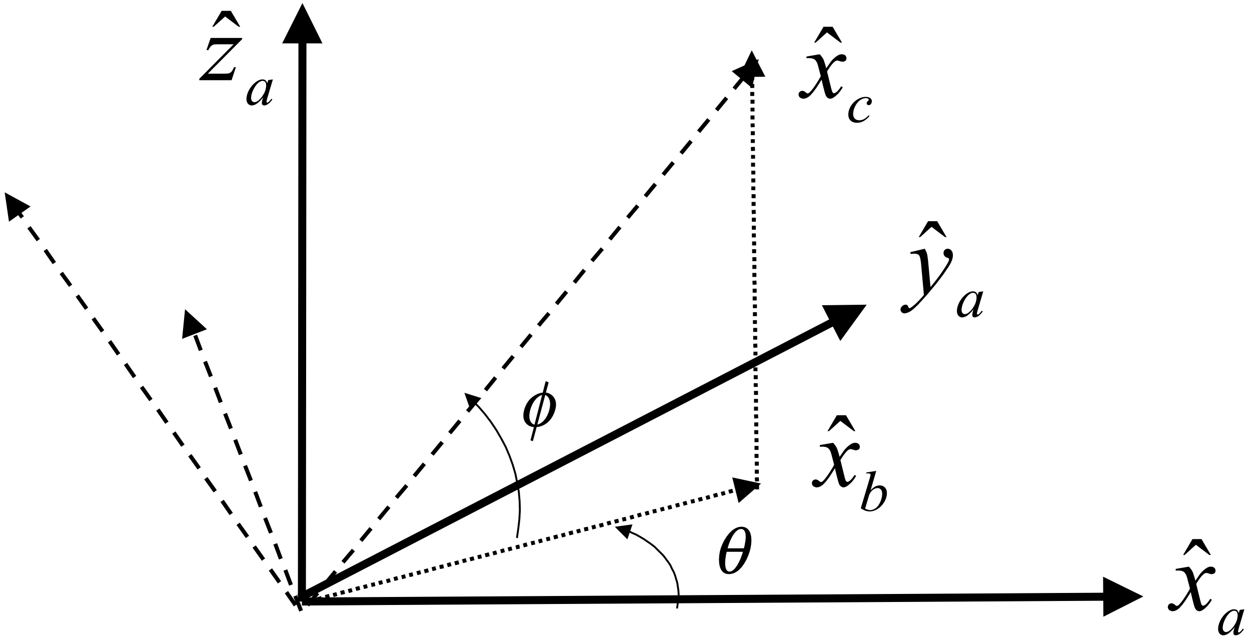
Example 2: composition of rotation matrices



$$^{a}R_{b} = \begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$^{b}R_{c} = \begin{pmatrix} c_{-\phi} & 0 & s_{-\phi} \\ 0 & 1 & 0 \\ -s_{-\phi} & 0 & c_{-\phi} \end{pmatrix} = \begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix}$$

Example 2: composition of rotation matrices



$$ {}^aR_c={}^aR_b\,{}^bR_c=\begin{pmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_\phi & 0 & -s_\phi \\ 0 & 1 & 0 \\ s_\phi & 0 & c_\phi \end{pmatrix}=\begin{pmatrix} c_\theta c_\phi & -s_\theta & -c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & -s_\theta s_\phi \\ s_\phi & 0 & c_\phi \end{pmatrix} $$