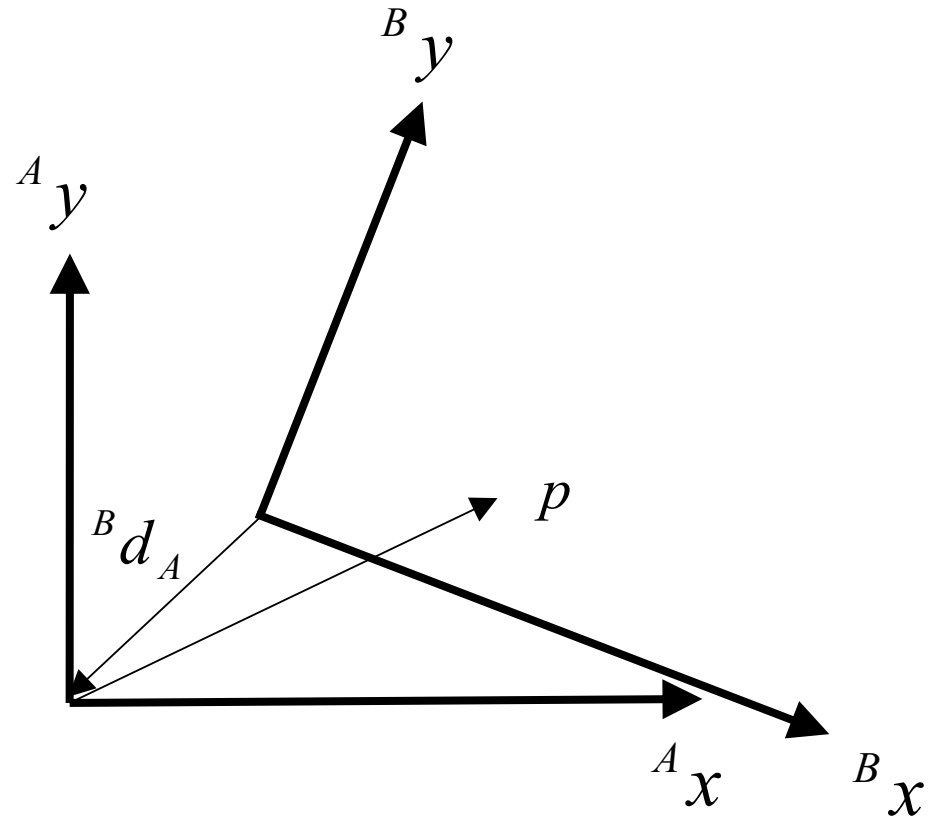
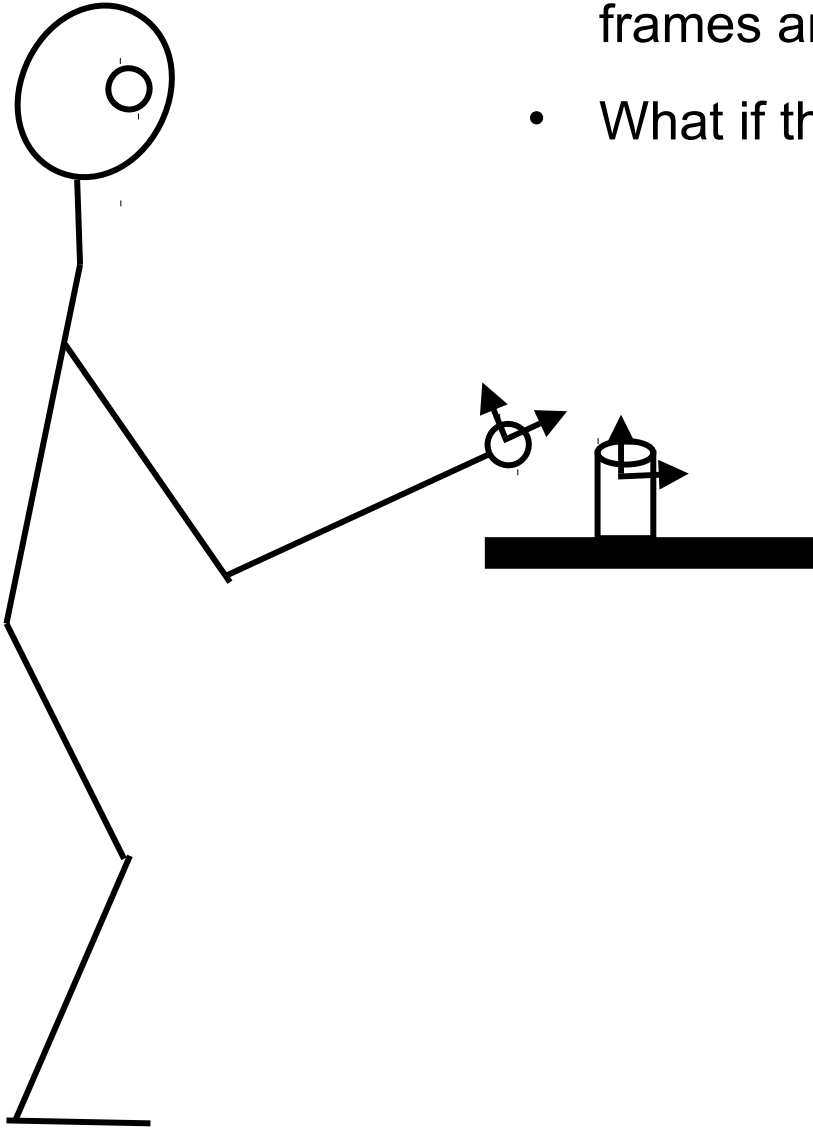


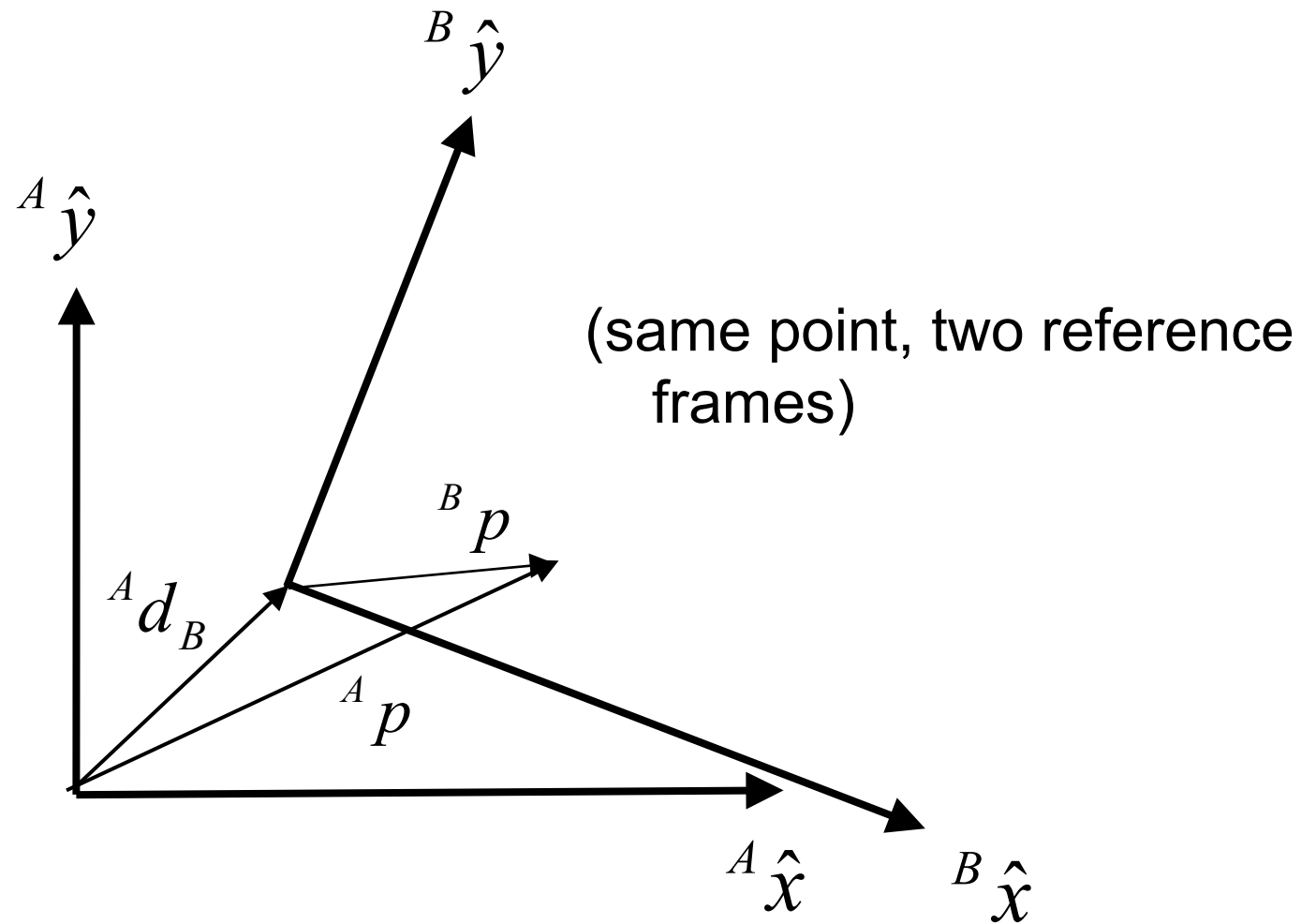
Homogeneous transforms

Rotation matrices assume that the origins of the two frames are co-located.

- What if they're separated by a translation?



Homogeneous transform



$$^A p = {}^A R_B {}^B p + {}^A d_B$$

Homogeneous transform

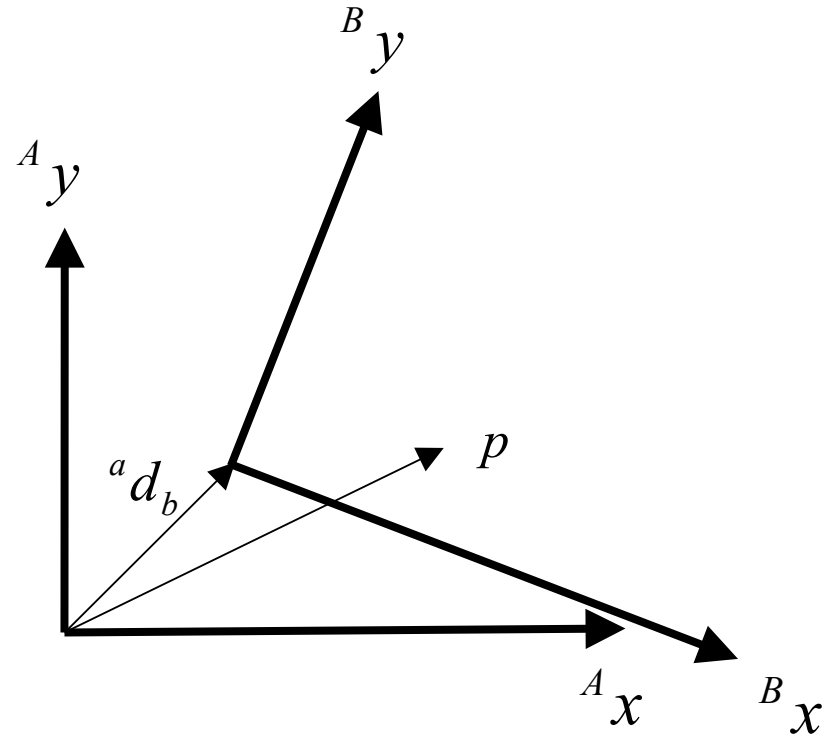
$${}^A p = {}^A R_B {}^B p + {}^A d_B$$

$$= \begin{pmatrix} {}^A R_B & {}^A d_B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^B p \\ 1 \end{pmatrix}$$

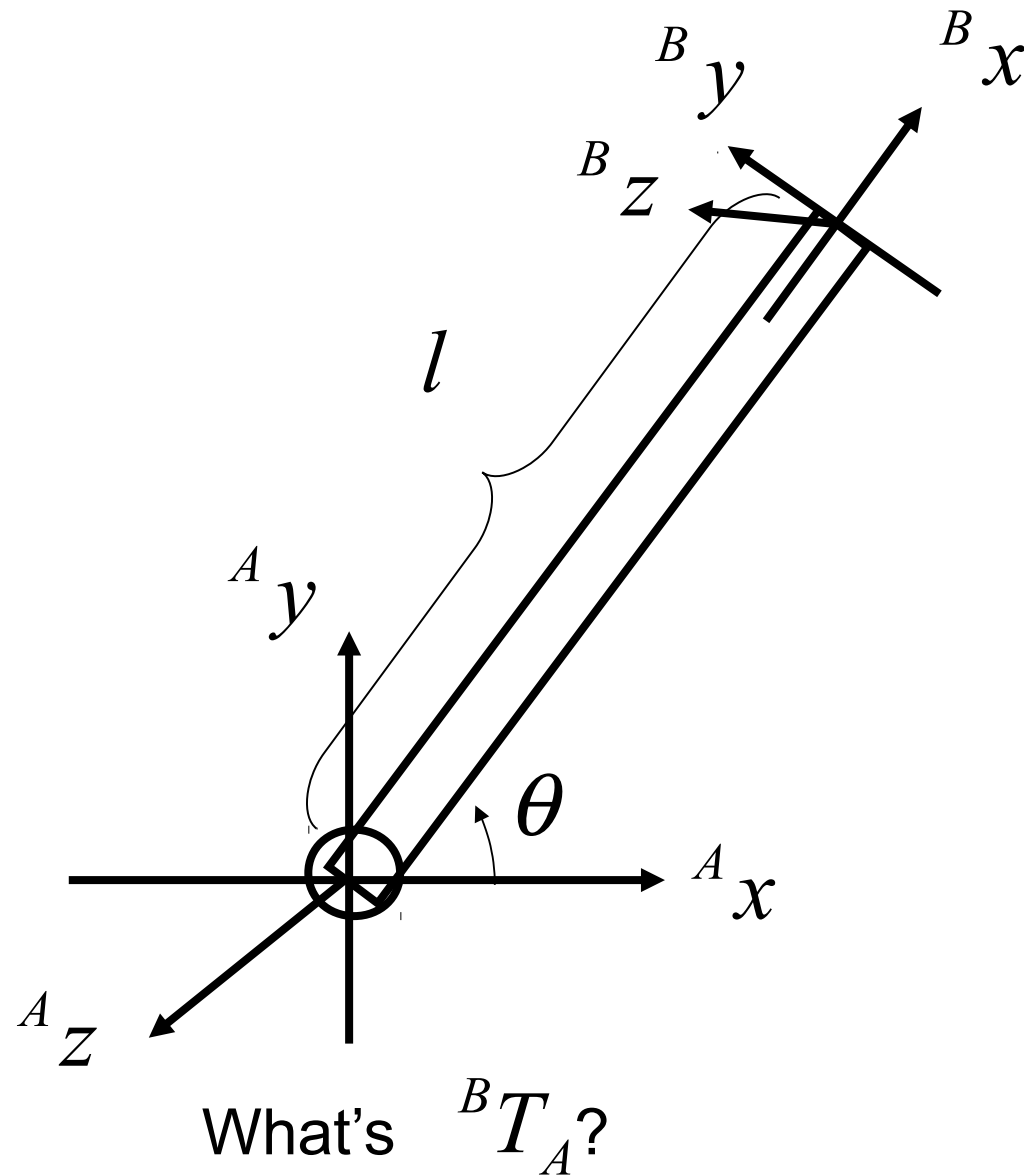
$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} & {}^A d_x \\ r_{21} & r_{22} & r_{23} & {}^A d_y \\ r_{31} & r_{32} & r_{33} & {}^A d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} {}^B p \\ 1 \end{pmatrix} = {}^A T_B \begin{pmatrix} {}^B p \\ 1 \end{pmatrix}$$

always zeros

always one



Example 1: homogeneous transforms



Example 1: homogeneous transforms

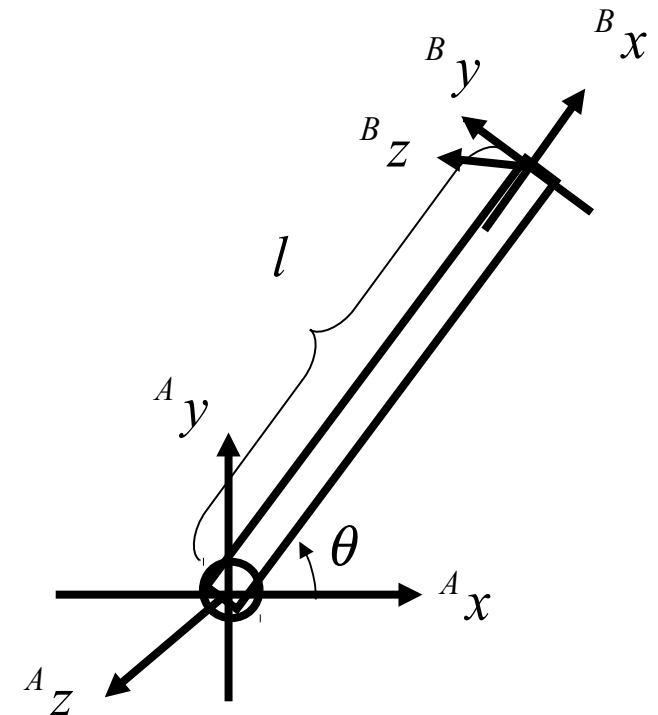
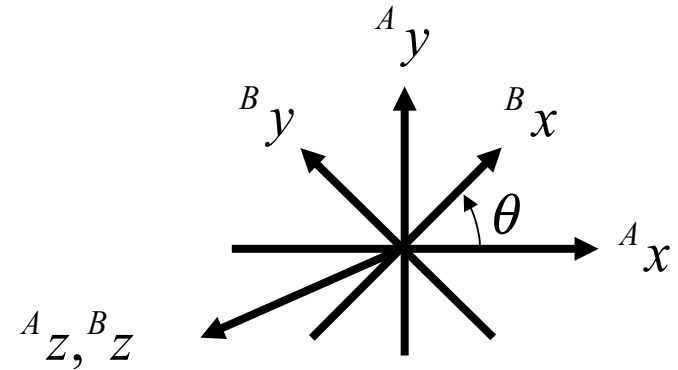
What's ${}^B T_A$?

$${}^A R_B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

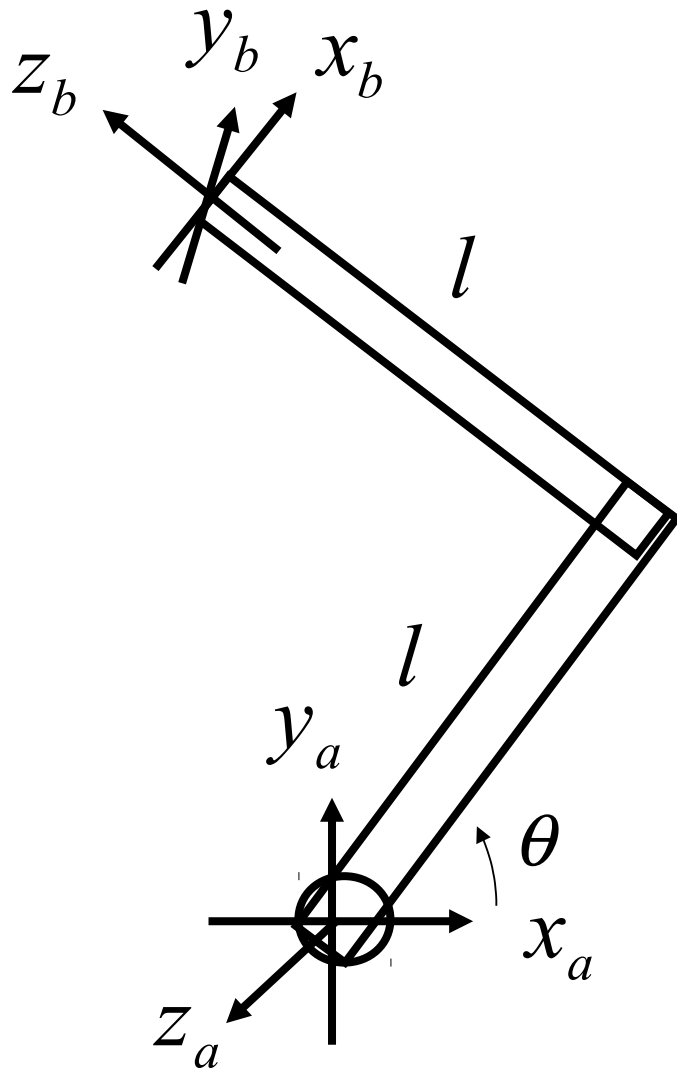
$${}^B d_A = \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix}$$

$${}^B T_A = \begin{pmatrix} {}^B R_A & {}^B d_A \\ 0 & 1 \end{pmatrix}$$

$${}^B T_A = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & -l \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Example 2: homogeneous transforms

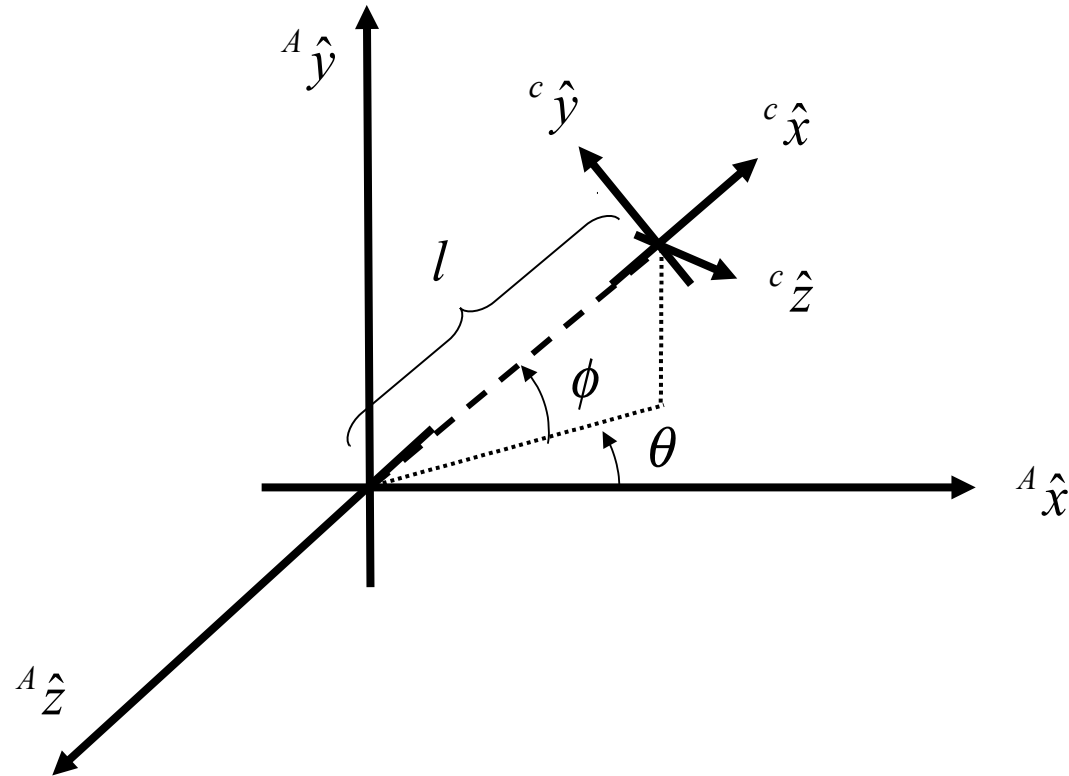


$${}^aT_b = \begin{bmatrix} c\theta & 0 & -s\theta & l\sqrt{2}c(\theta + \frac{\pi}{4}) \\ s\theta & 0 & c\theta & l\sqrt{2}s(\theta + \frac{\pi}{4}) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This arm rotates about the z_a axis.

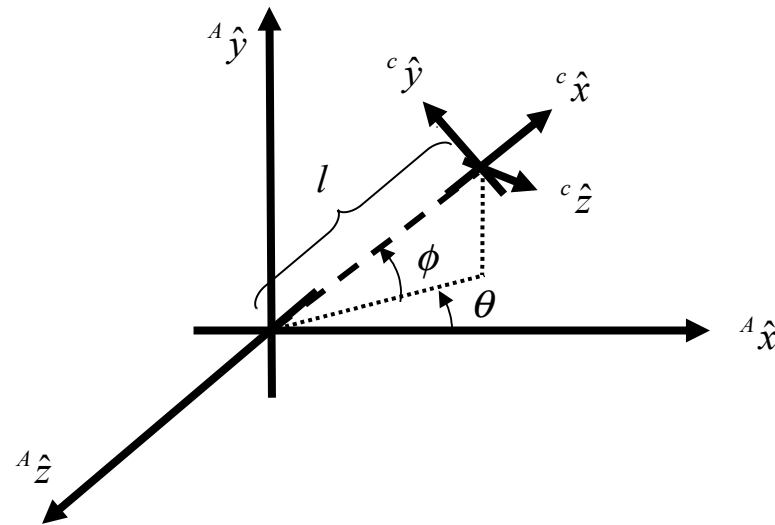
What's aT_b ?

Example 3: homogeneous transforms



$${}^aR_c = {}^aR_b {}^bR_c = \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix} \begin{pmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c_\theta c_\phi & -s_\phi c_\theta & s_\theta \\ s_\phi & c_\phi & 0 \\ -s_\theta c_\phi & s_\theta s_\phi & c_\theta \end{pmatrix}$$

Example 3: homogeneous transforms



$${}^c d = \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix} \quad {}^a d = -{}^a R_c {}^c d = - \begin{pmatrix} c_\theta c_\phi & -s_\phi c_\theta & s_\theta \\ s_\phi & c_\phi & 0 \\ -s_\theta c_\phi & s_\theta s_\phi & c_\theta \end{pmatrix} \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l c_\theta c_\phi \\ l s_\phi \\ -l s_\theta c_\phi \end{pmatrix}$$

$${}^a T_c = \begin{pmatrix} {}^a R_c & {}^a d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c_\theta c_\phi & -s_\phi c_\theta & s_\theta & l c_\theta c_\phi \\ s_\phi & c_\phi & 0 & l s_\phi \\ -s_\theta c_\phi & s_\theta s_\phi & c_\theta & -l s_\theta c_\phi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inverse of the Homogeneous transform

Can also derive it from the forward Homogeneous transform:

$${}^B p = {}^B R_A {}^A p + {}^B d_A$$

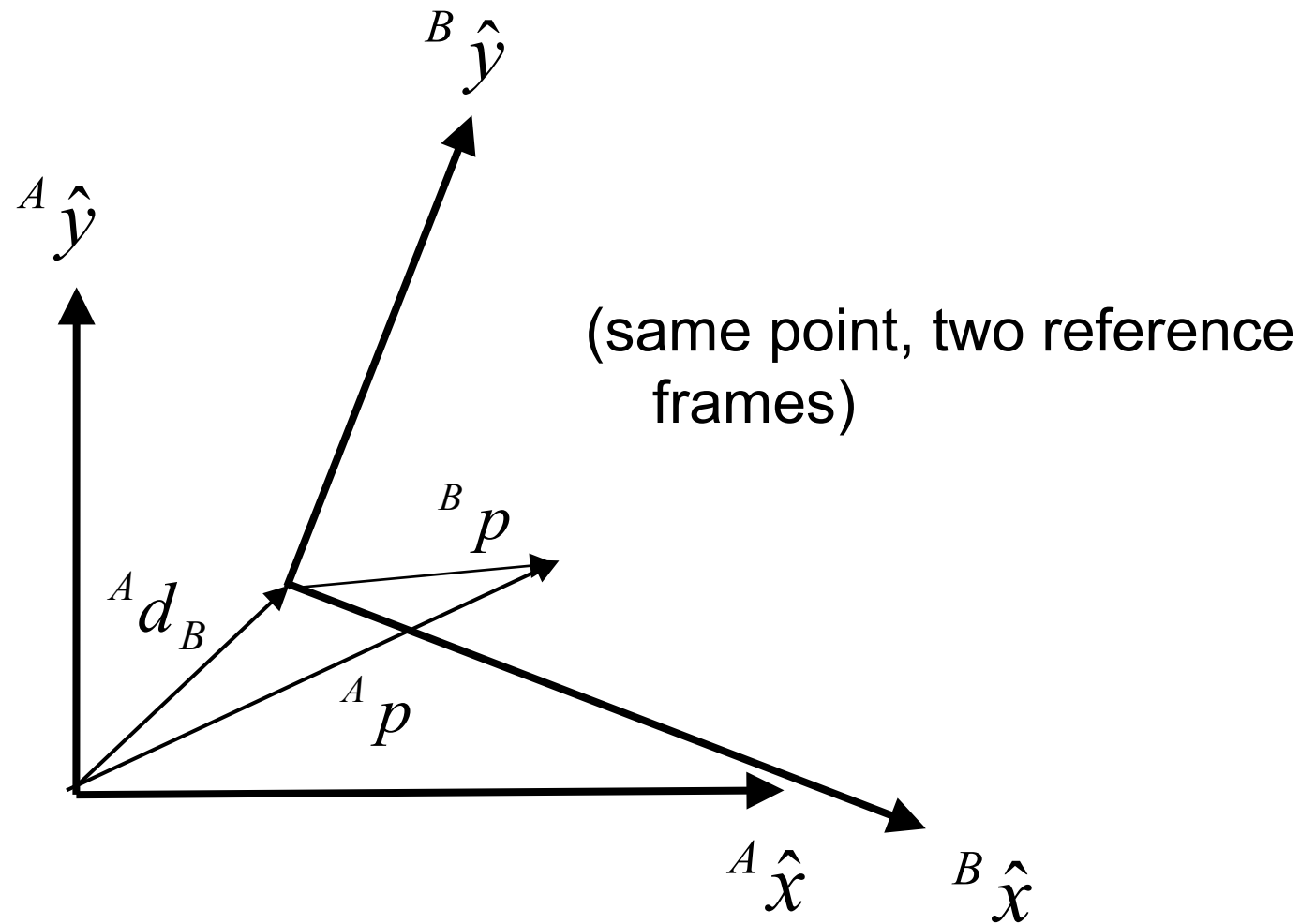
$${}^A p = {}^B R_A^T ({}^B p - {}^B d_A)$$

$${}^A p = {}^B T_A^{-1} \begin{pmatrix} {}^B p \\ 1 \end{pmatrix}$$

where

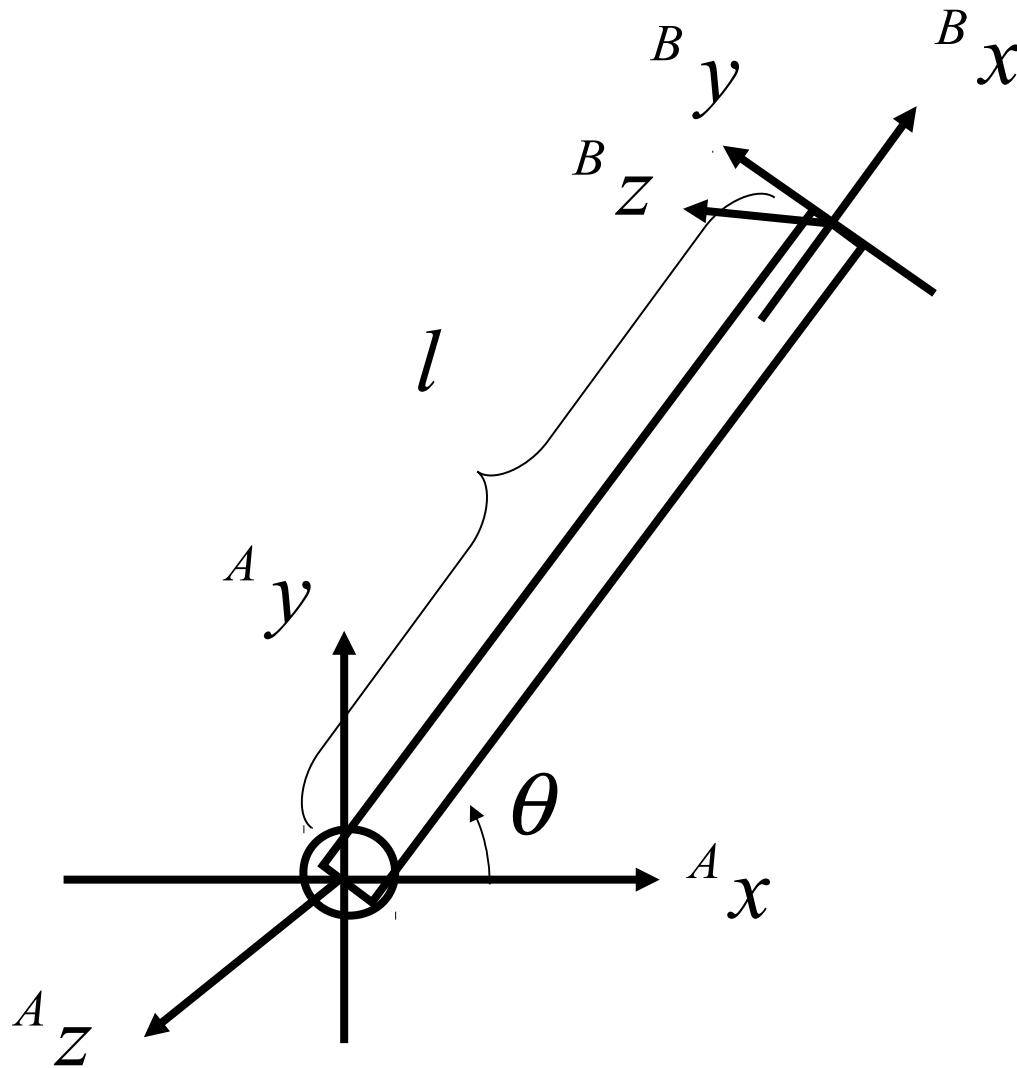
$${}^B T_A^{-1} = \begin{pmatrix} {}^B R_A^T & -{}^B R_A^T {}^B d_A \\ 0 & 1 \end{pmatrix}$$

Inverse of the Homogeneous transform



$$^B p = {}^B R_A {}^A p - {}^B R_A {}^A d_B = {}^B R_A ({}^A p - {}^A d_B)$$

Example 1: homogeneous transform inverse



$${}^B T_A = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & -l \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

What's ${}^A T_B$?

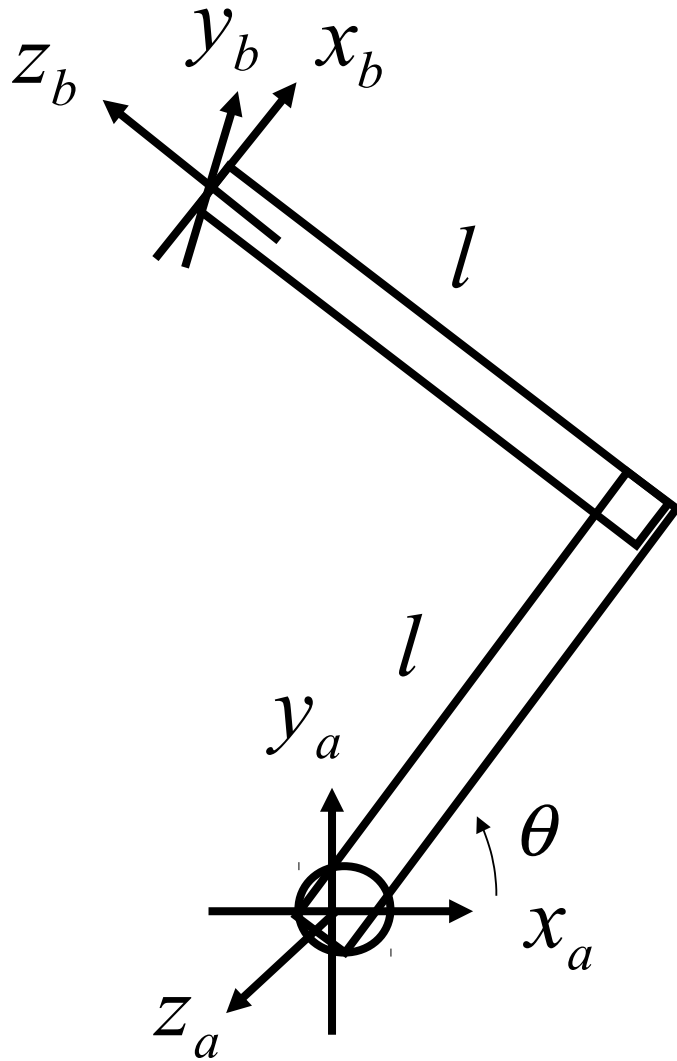
Example 1: homogeneous transform inverse

$${}^B T_A^{-1} = \begin{pmatrix} {}^B R_A^T & -{}^B R_A^T {}^B d_A \\ 0 & 1 \end{pmatrix} \quad {}^A R_B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}^B d_A = \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix} \quad -{}^A R_B {}^B d_A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -l \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} l \cos(\theta) \\ l \sin(\theta) \\ 0 \end{pmatrix}$$

$${}^B T_A^{-1} = {}^A T_B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 & l \cos(\theta) \\ \sin(\theta) & \cos(\theta) & 0 & l \sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example 2: homogeneous transform inverse

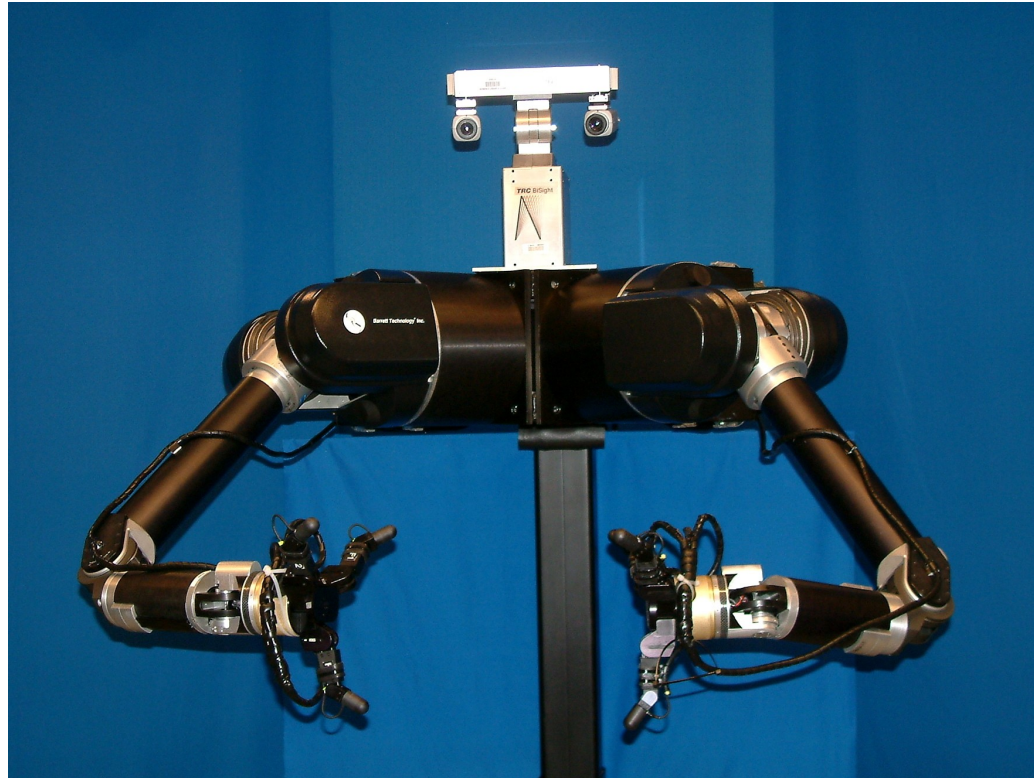


$${}^aT_b = \begin{bmatrix} c\theta & 0 & -s\theta & l\sqrt{2}c(\theta + \frac{\pi}{4}) \\ s\theta & 0 & c\theta & l\sqrt{2}s(\theta + \frac{\pi}{4}) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What's bT_a ?

$${}^bT_a = \begin{bmatrix} c\theta & s\theta & 0 & -l\sqrt{2}(c_\theta c_{\theta+\frac{\pi}{4}} + s_\theta s_{\theta+\frac{\pi}{4}}) \\ 0 & 0 & -1 & 0 \\ -s\theta & c\theta & 0 & l\sqrt{2}(s_\theta c_{\theta+\frac{\pi}{4}} - c_\theta s_{\theta+\frac{\pi}{4}}) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics



- Where is the end effector w.r.t. the “base” frame?

Composition of homogeneous transforms

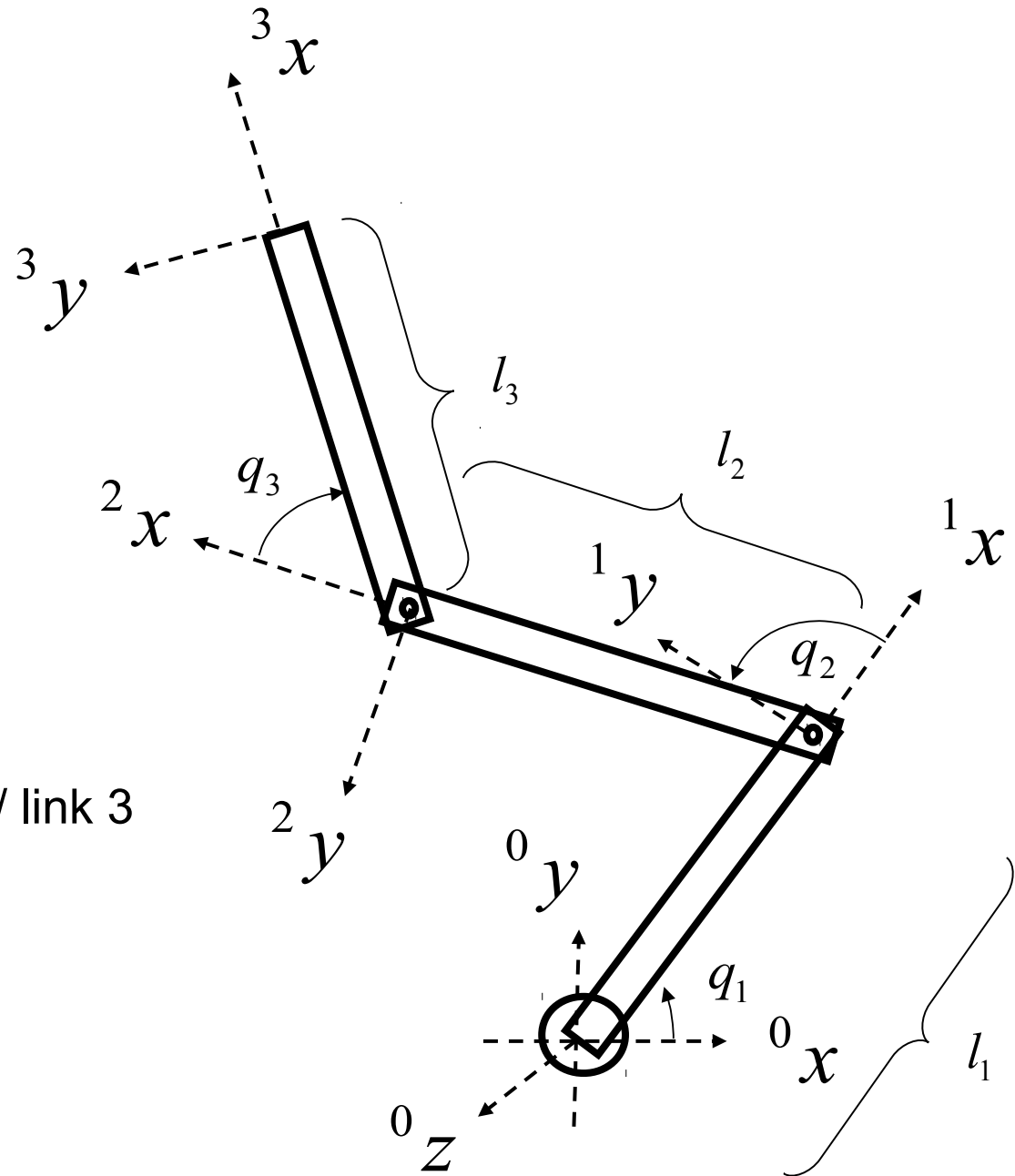
Base to *eff* transform

$${}^0T_3 = {}^0T_1 {}^1T_2 {}^2T_3$$

Transform associated w/ link 1

Transform associated w/ link 2

Transform associated w/ link 3

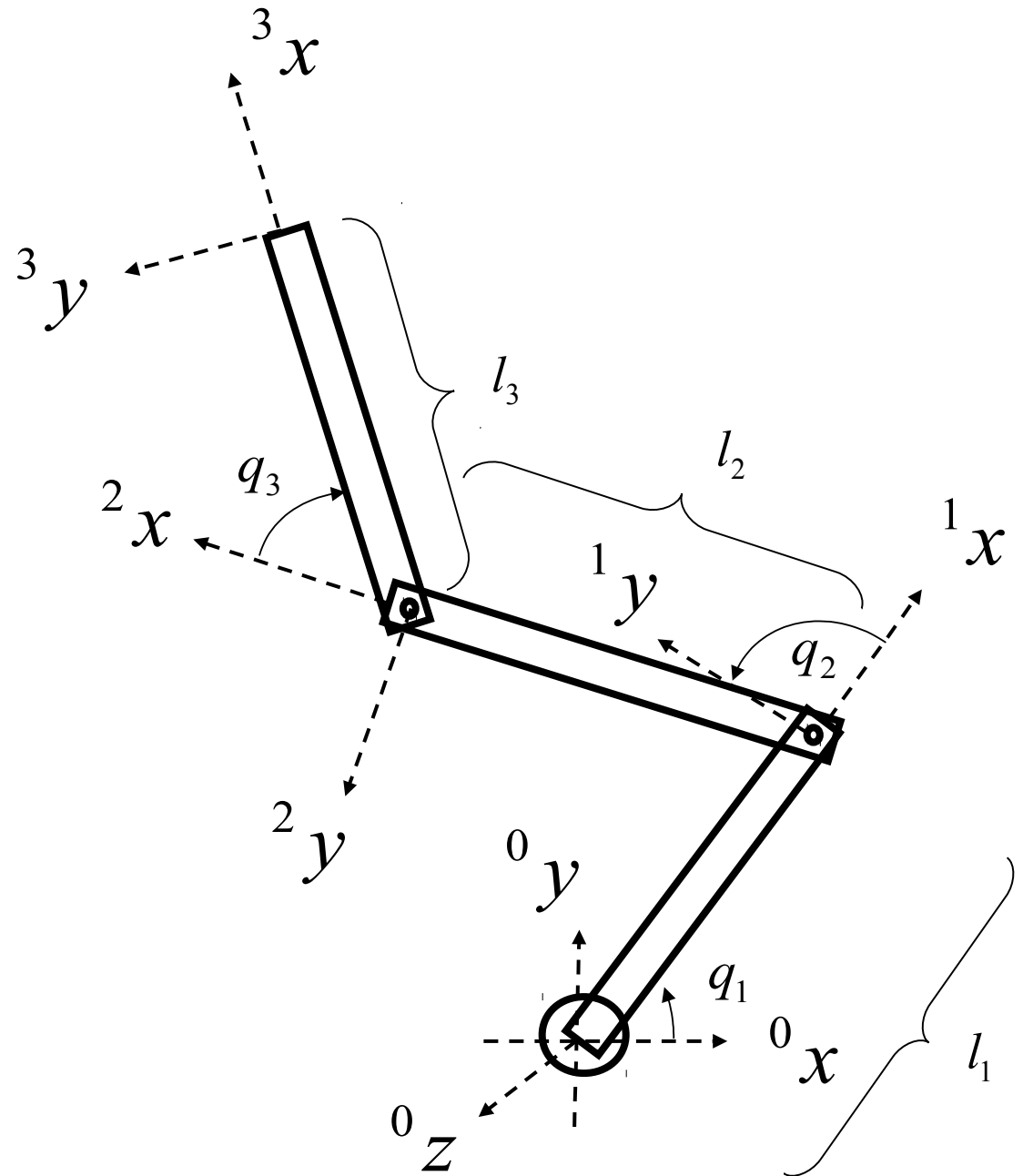


Forward kinematics: composition of homogeneous transforms

$${}^0T_3 = {}^0T_1 {}^1T_2 {}^2T_3$$

$${}^0T_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

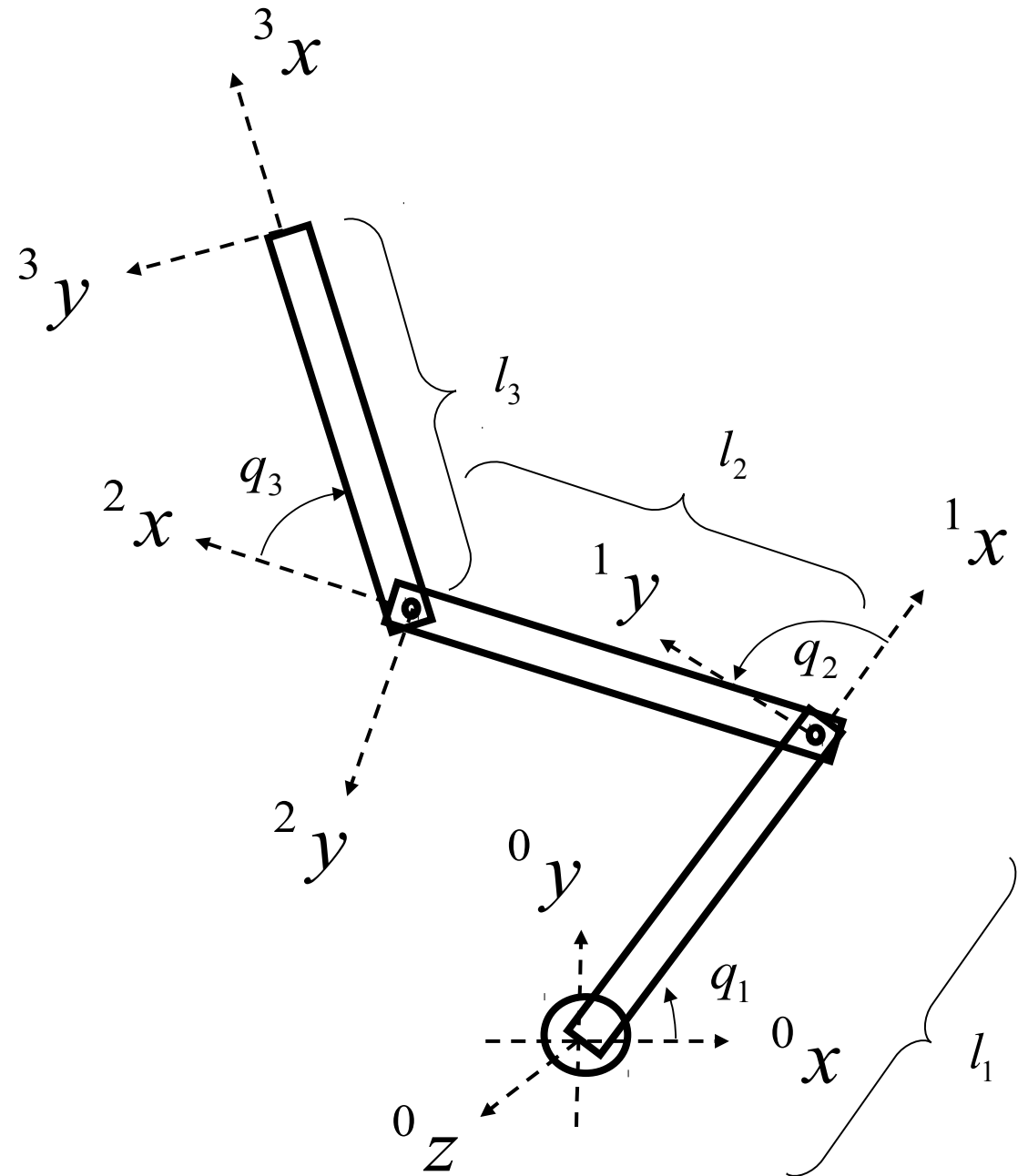
$${}^1T_2 = \begin{pmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Forward kinematics: composition of homogeneous transforms

$${}^0T_3 = {}^0T_1 {}^1T_2 {}^2T_3$$

$${}^2T_3 = \begin{pmatrix} c_3 & -s_3 & 0 & l_3 c_3 \\ s_3 & c_3 & 0 & l_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Remember those double-angle formulas...

$$\sin(\theta \pm \phi) = \sin(\theta) \cos(\phi) \pm \cos(\theta) \sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta) \cos(\phi) \mp \sin(\theta) \sin(\phi)$$

Forward kinematics: composition of homogeneous transforms

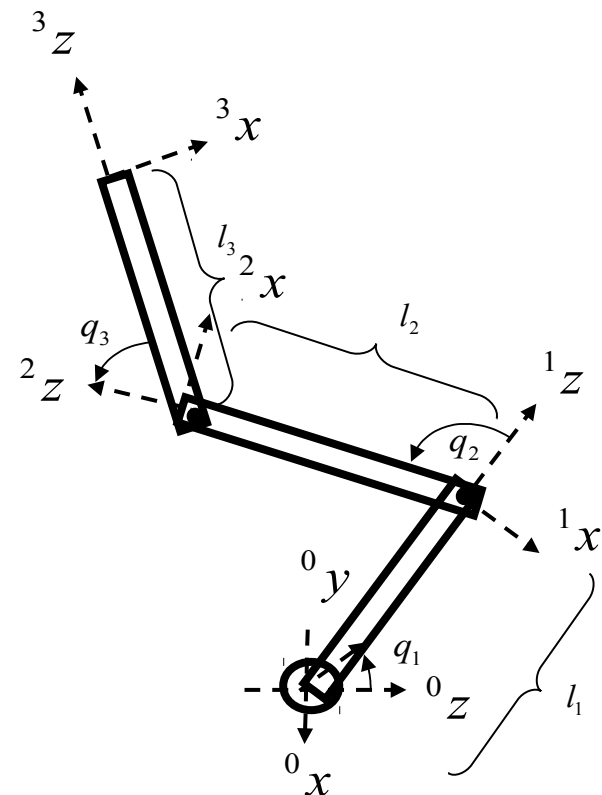
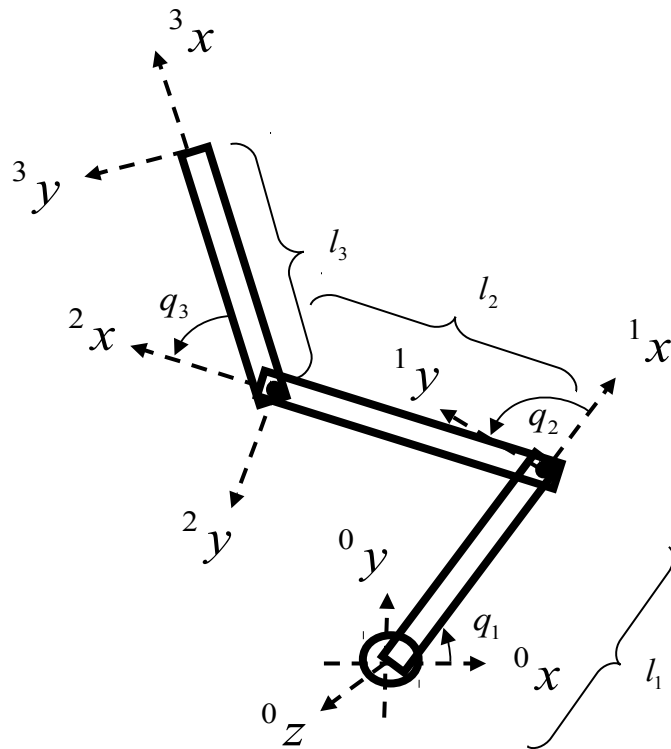
$${}^0T_3 = {}^0T_1 {}^1T_2 {}^2T_3$$

$${}^0T_3 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1 c_1 \\ s_1 & c_1 & 0 & l_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_2 & -s_2 & 0 & l_2 c_2 \\ s_2 & c_2 & 0 & l_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_3 & -s_3 & 0 & l_3 c_3 \\ s_3 & c_3 & 0 & l_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0T_3 = \begin{pmatrix} c_{123} & -s_{123} & 0 & l_1 c_1 + l_2 c_{12} + l_3 c_{123} \\ s_{123} & c_{123} & 0 & l_1 s_1 + l_2 s_{12} + l_3 s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

DH parameters

- There are a large number of ways that homogeneous transforms can encode the kinematics of a manipulator
- We will sacrifice some of this flexibility for a more systematic approach: DH (Denavit-Hartenberg) parameters.
- DH parameters is a standard for describing a series of transforms for arbitrary mechanisms.



Forward kinematics: DH parameters

These four DH parameters,

$$(a_i \quad \alpha_i \quad d_i \quad \theta_i)$$

represent the following homogeneous matrix:

$$T = \underbrace{\begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{First, translate by } d_i \text{ along z axis and rotate by } \theta_i \text{ about z axis}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Then, translate by } a_i \text{ along x axis and rotate by } \alpha_i \text{ about x axis}}$$

First, translate by d_i along z axis
and rotate by θ_i about z axis

Then, translate by a_i along x axis
and rotate by α_i about x axis

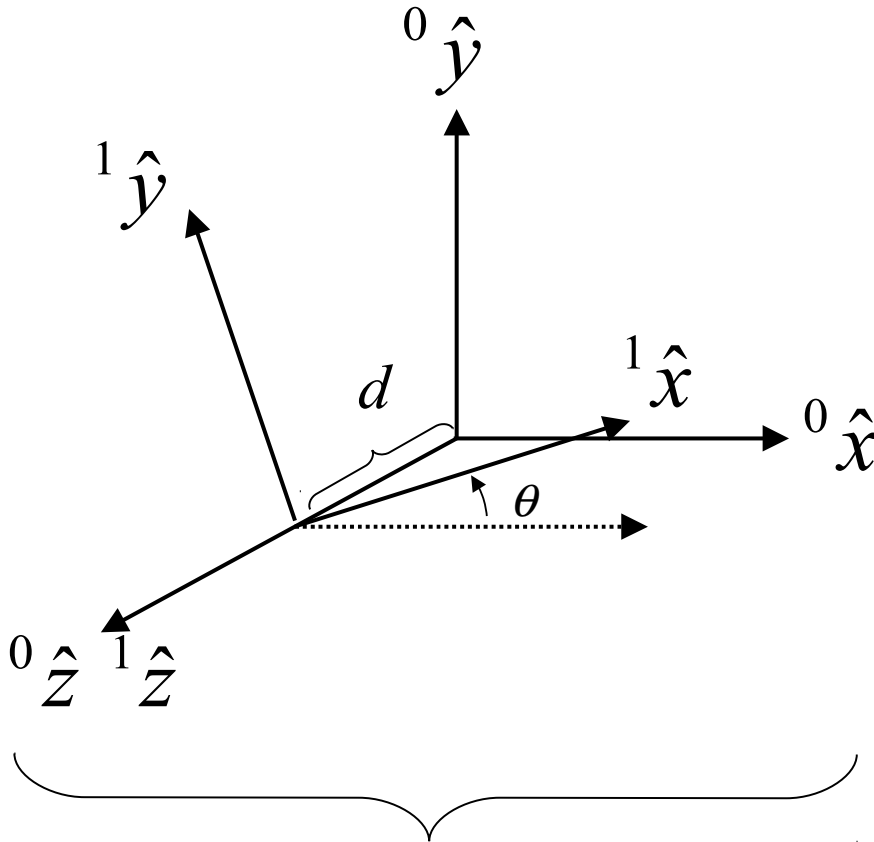
Forward kinematics: DH parameters

$$\begin{aligned}
 T &= \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} c_{\theta_i} & -s_{\theta_i} c_{\alpha_i} & s_{\theta_i} s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i} c_{\alpha_i} & -c_{\theta_i} s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

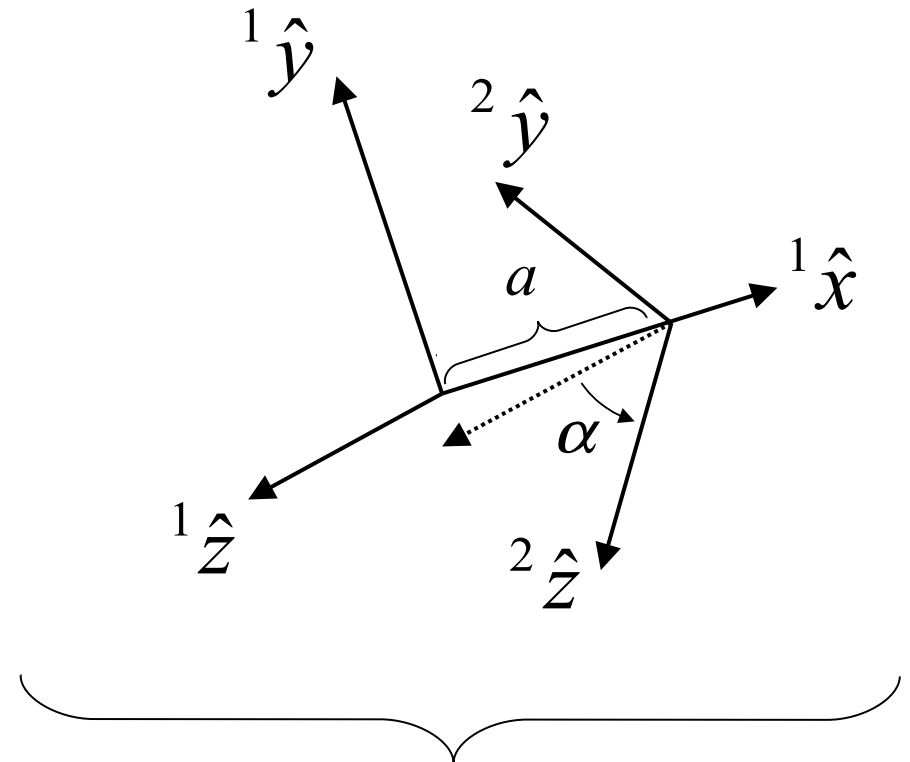
Forward kinematics: DH parameters

Four DH parameters: $(a_i \quad \alpha_i \quad d_i \quad \theta_i)$

$$T = T_{rot(z, \theta_i)} T_{trans(z, d_i)} T_{rot(x, \alpha_i)} T_{trans(x, a_i)}$$



First, translate by d_i along z axis
and rotate by θ_i about z axis



Then, translate by a_i along x axis
and rotate by α_i about x axis

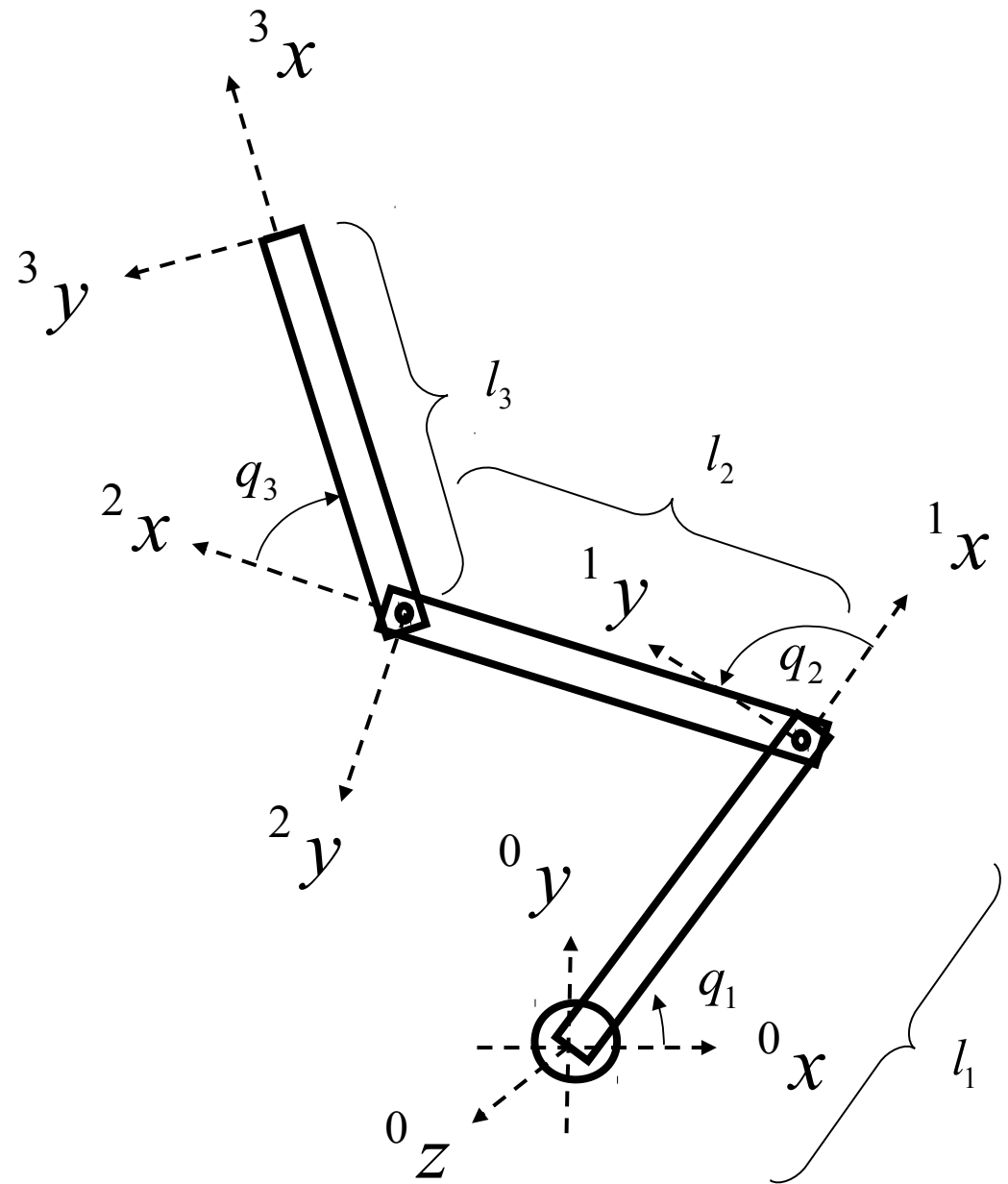
Forward kinematics: DH parameters

- A series of transforms is written as a table:

xform	a_i	α_i	d_i	θ_i
1	a_1	α_1	d_1	θ_1
2	a_2	α_2	d_2	θ_2

Example 1: DH parameters

	a_i	α_i	d_i	θ_i
1	l_1	0	0	q_1
2	l_2	0	0	q_2
3	l_3	0	0	q_3



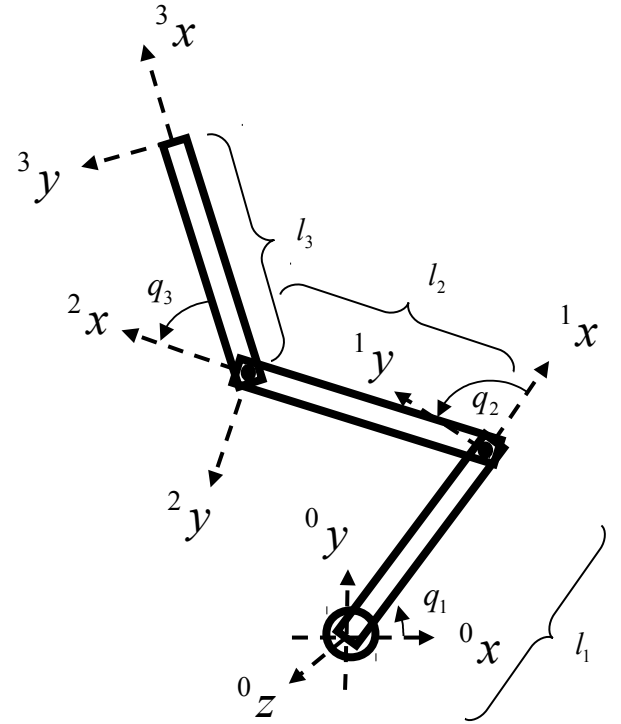
Example 1: DH parameters

$${}^0T_1 = \begin{pmatrix} c_{q_1} & -s_{q_1} & 0 & l_1 c_{q_1} \\ s_{q_1} & c_{q_1} & 0 & l_1 s_{q_1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^1T_2 = \begin{pmatrix} c_{q_2} & -s_{q_2} & 0 & l_2 c_{q_2} \\ s_{q_2} & c_{q_2} & 0 & l_2 s_{q_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

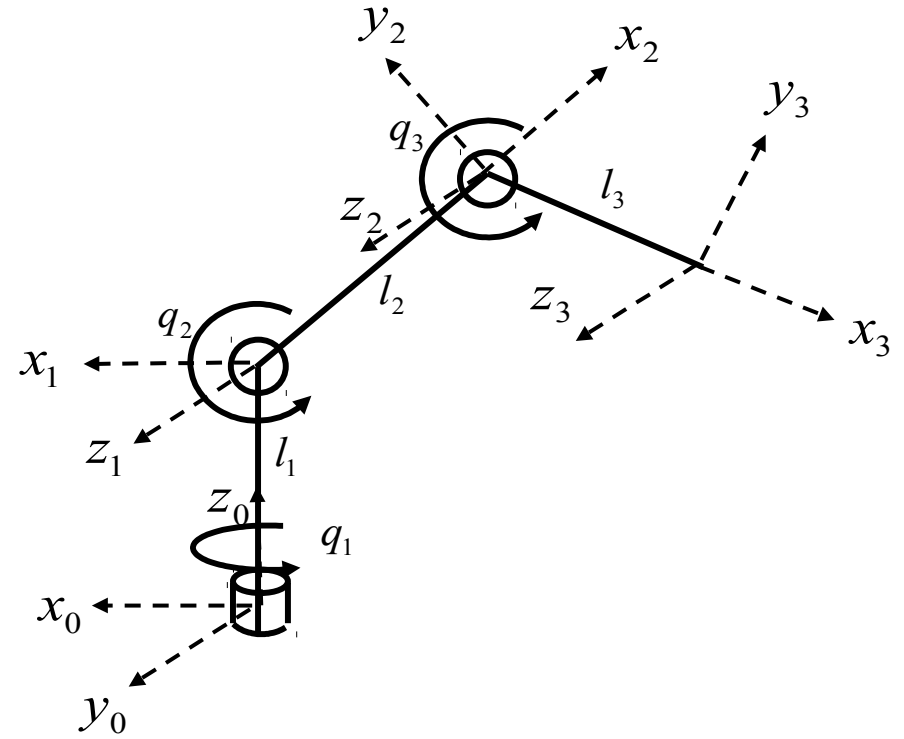
$${}^2T_3 = \begin{pmatrix} c_{q_3} & -s_{q_3} & 0 & l_3 c_{q_3} \\ s_{q_3} & c_{q_3} & 0 & l_3 s_{q_3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$${}^0T_3 = {}^0T_1 {}^1T_2 {}^2T_3$$

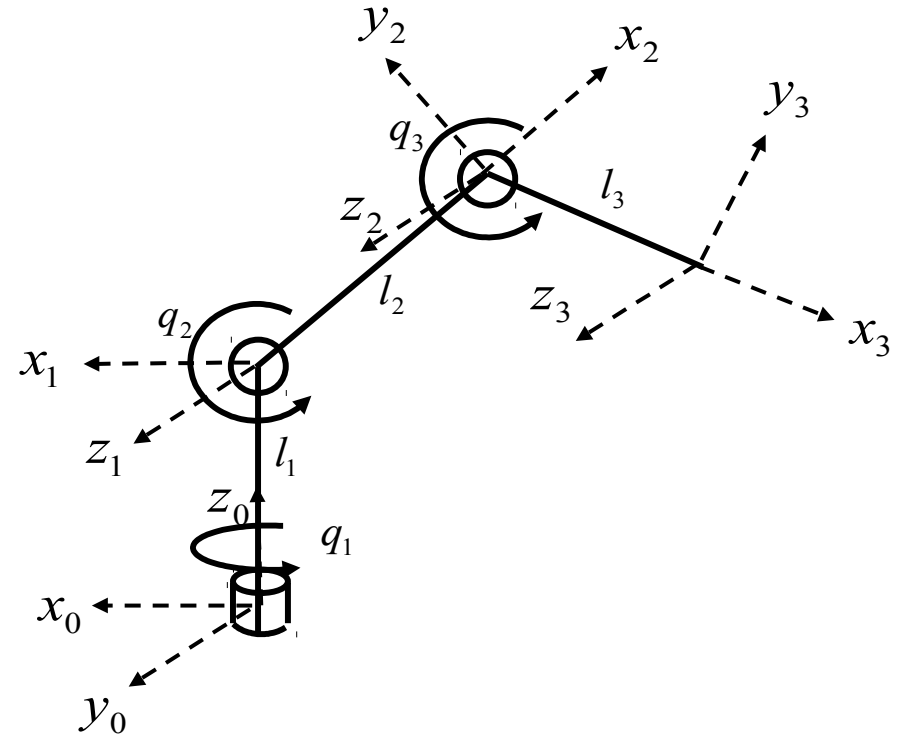


	a_i	α_i	d_i	θ_i
1	l_1	0	0	q_1
2	l_2	0	0	q_2
3	l_3	0	0	q_3

Example 2: DH parameters

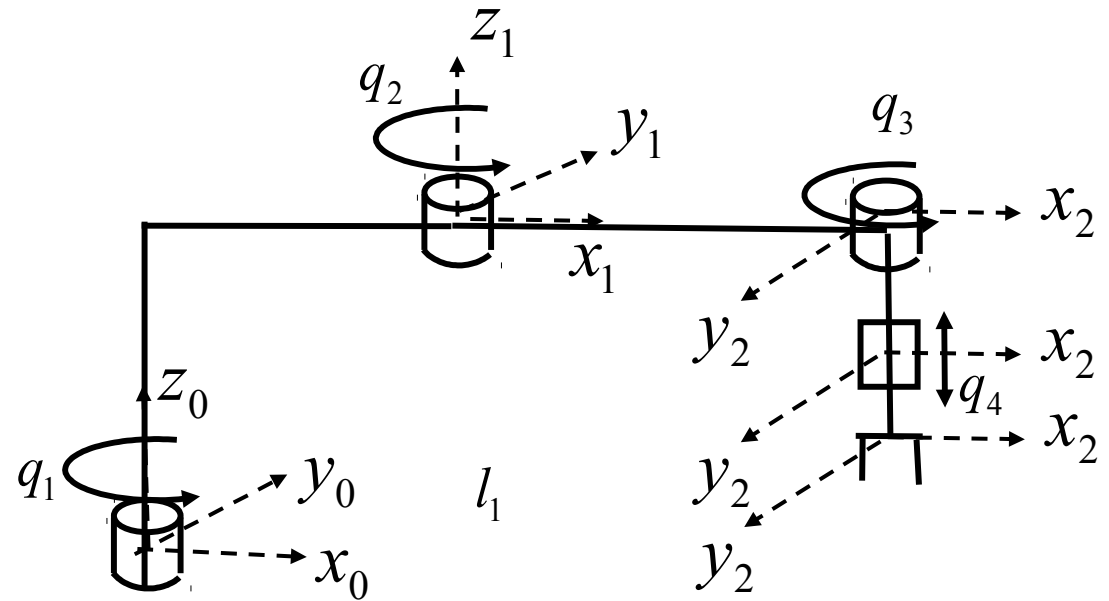


Example 2: DH parameters



	a_i	α_i	d_i	θ_i
1	0	$-\frac{\pi}{2}$	l_1	q_1
2	l_2	0	0	$q_2 - \frac{\pi}{2}$
3	l_3	0	0	q_3

Example 3: DH parameters



Example 3: DH parameters

	a_i	α_i	d_i	θ_i
1	l_2	0	l_1	q_1
2	l_3	π	0	q_2
3	0	0	l_4	q_3
4	0	0	q_4	0

