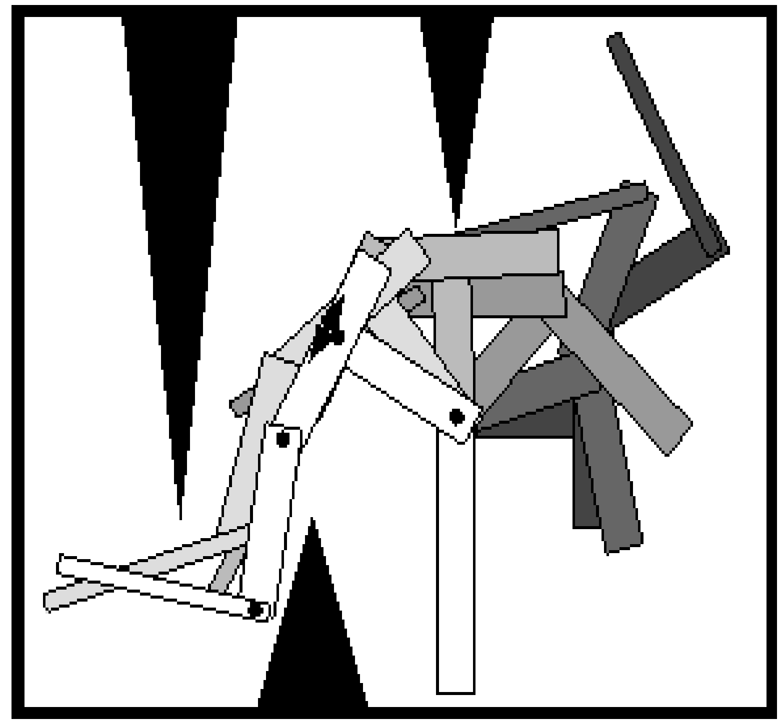
Configuration Space

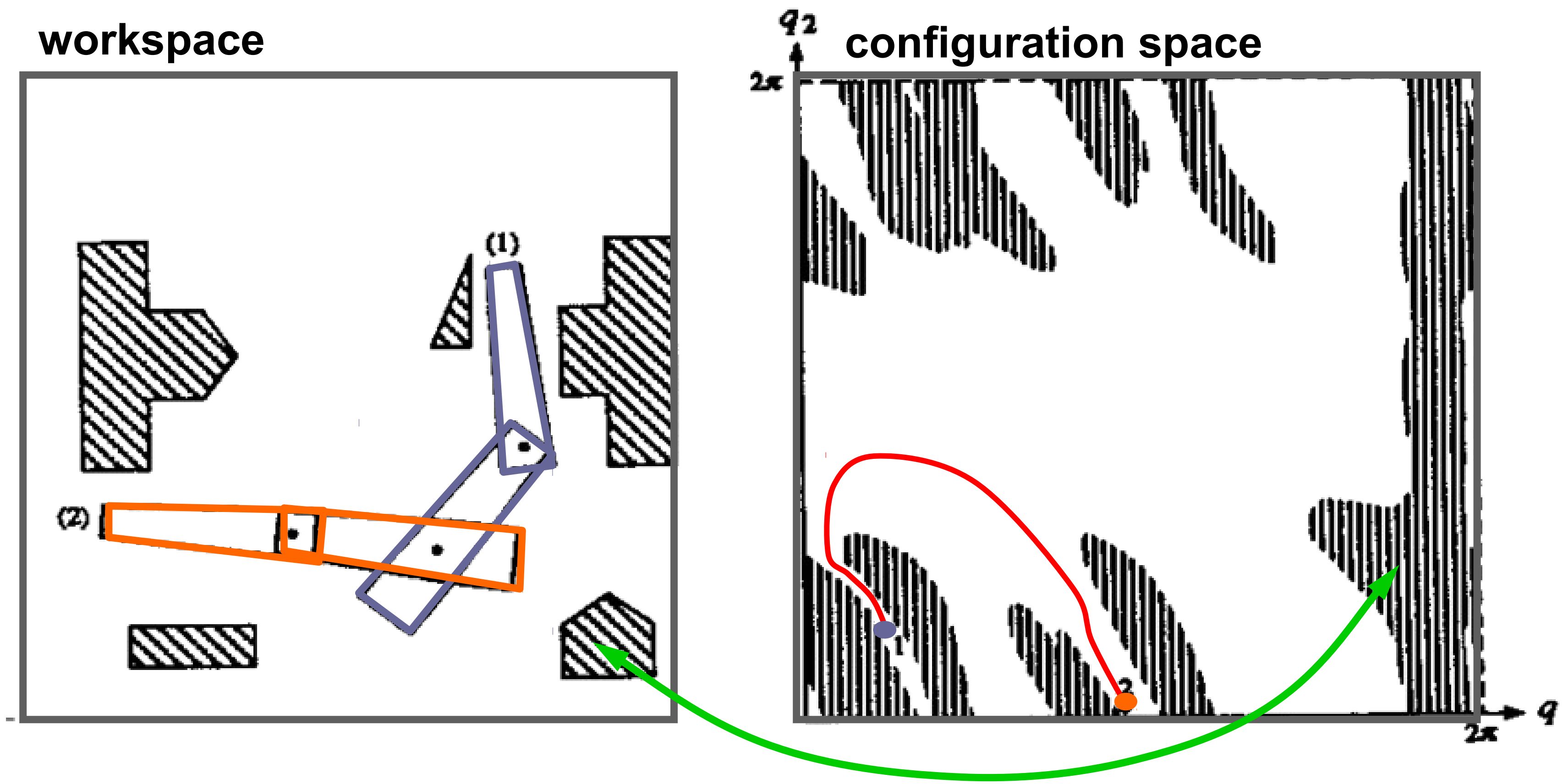
What is a path?



Rough idea

- Convert rigid robots, articulated robots, *etc.* into points
- Apply algorithms for moving points

Mapping from the workspace to the configuration space

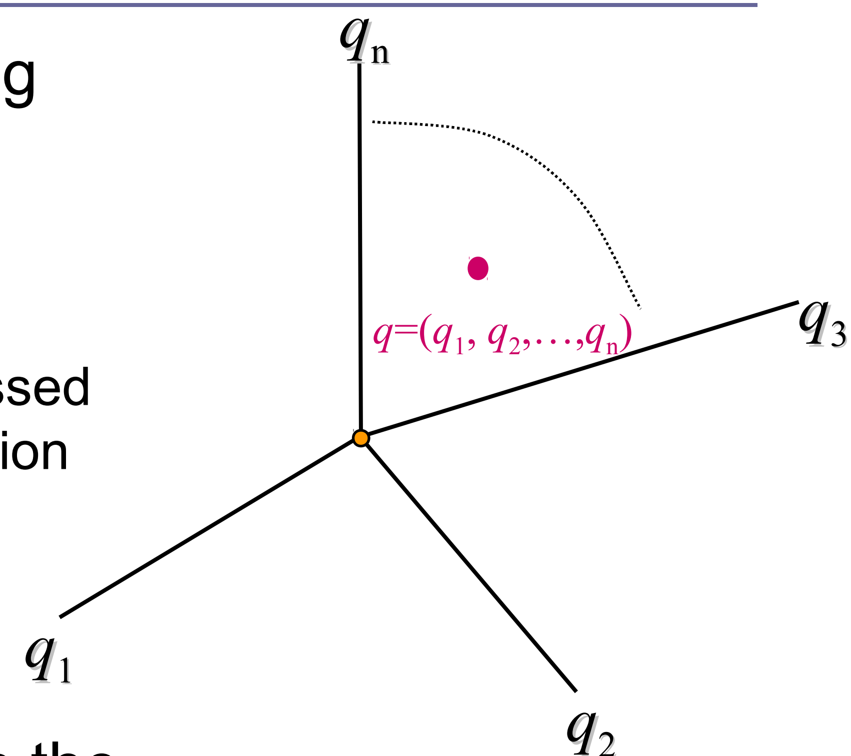


Configuration space

- **Definitions and examples**
- Obstacles
- Paths
- Metrics

Configuration space

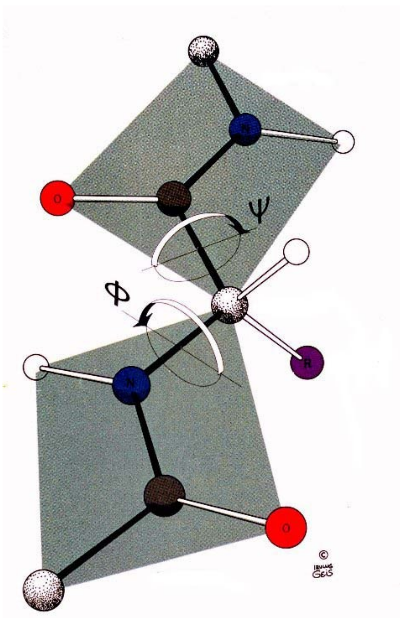
- The **configuration** of a moving object is a specification of the position of **every** point on the object.
  - Usually a configuration is expressed as a vector of position & orientation parameters: $q = (q_1, q_2, \ldots, q_n)$.

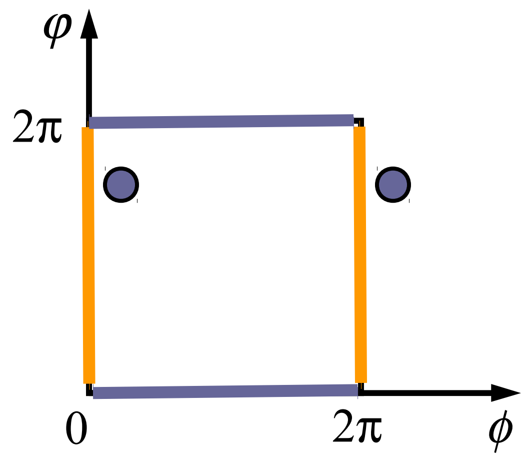


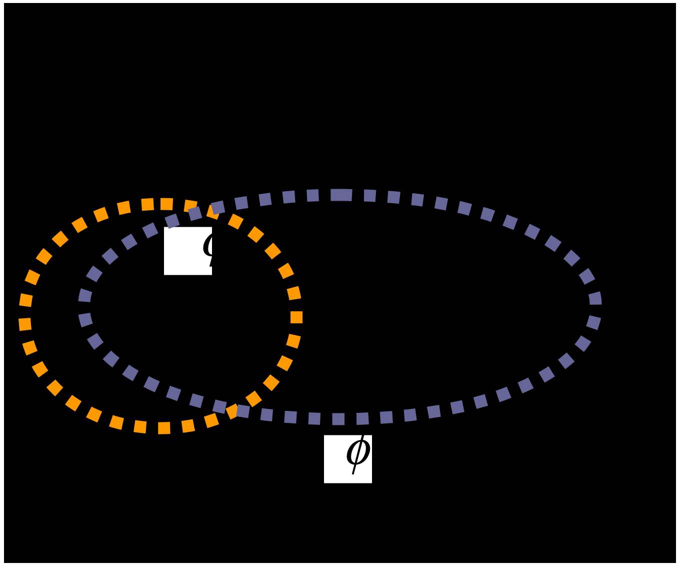
- The **configuration space** C is the set of all possible configurations.
  - A configuration is a point in C.

Topology of the configuration pace

- The topology of C is usually **not** that of a Cartesian space R^n.



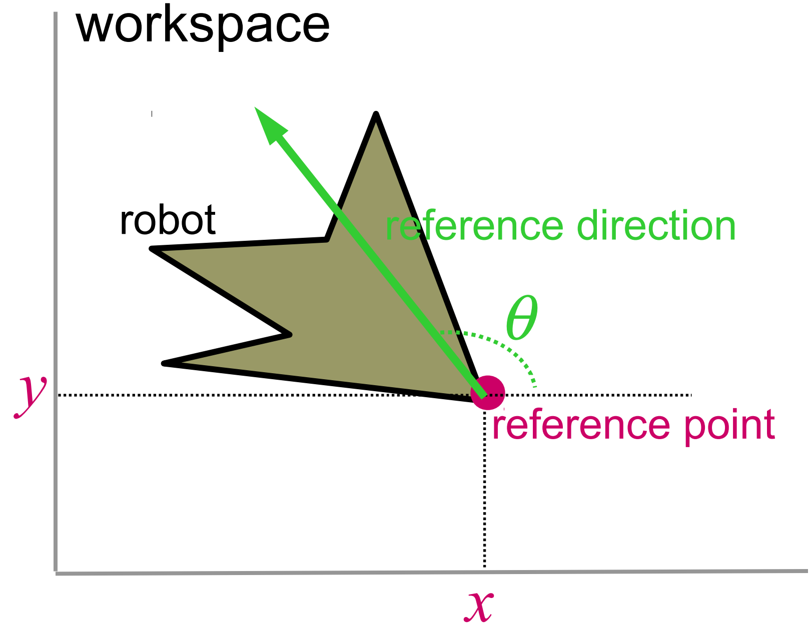




Dimension of configuration space

- The **dimension of a configuration space** is the **minimum** number of parameters needed to specify the configuration of the object completely.
- It is also called the **number of degrees of freedom** (dofs) of a moving object.

Example: rigid robot in 2-D workspace

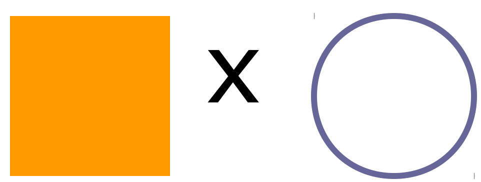


- 3-parameter specification: $q = (x, y, \theta)$ with $\theta \in [0, 2\pi)$.
  - 3-D configuration space

Example: rigid robot in 2-D workspace

- 4-parameter specification: $q = (x, y, u, v)$ with $u^2+v^2 = 1$. Note $u = \cos\theta$ and $v = \sin\theta$.

- dim of configuration space = **3**
  - Does the dimension of the configuration space (number of dofs) depend on the parametrization?
- Topology: a 3-D cylinder $C = \mathrm{R}^2 \text{x } \mathrm{S}^1$



  - Does the topology depend on the parametrization?

Example: rigid robot in 3-D workspace

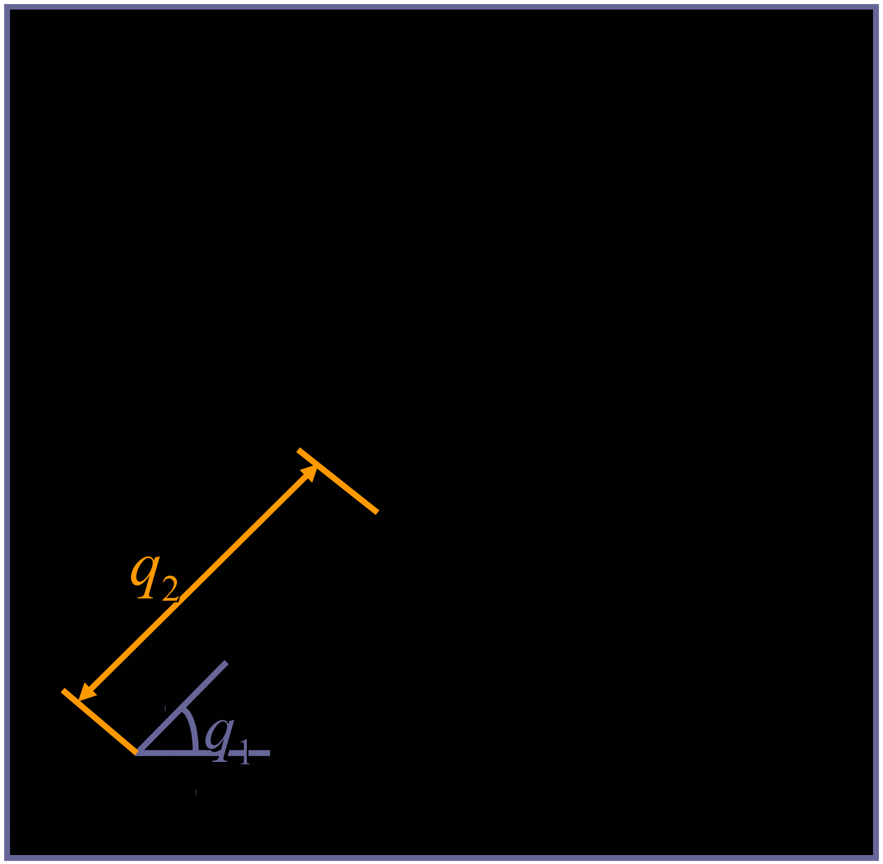
- q = (position, orientation) = $(x, y, z, ???)$

- Parametrization of orientations by matrix:
  $q = (r_{11}, r_{12}, \ldots, r_{33}, r_{33})$ where $r_{11}, r_{12}, \ldots, r_{33}$ are the elements of rotation matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

  with
  - $r_{1i}^2 + r_{2i}^2 + r_{3i}^2 = 1$ for all i ,
  - $r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0$ for all $i \neq j$,
  - $\det(R) = +1$

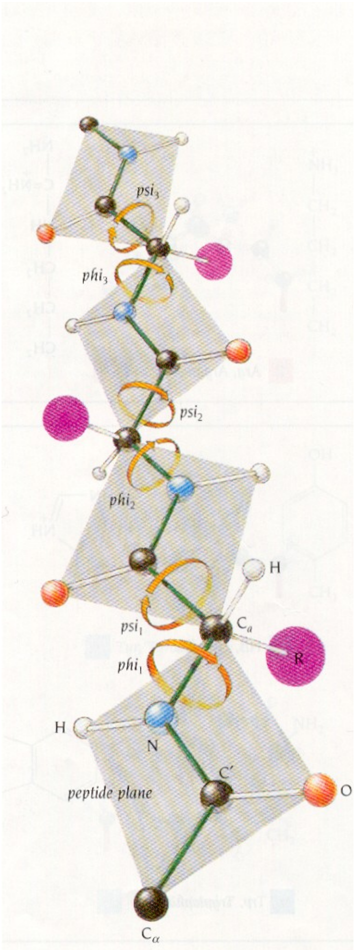
Example: articulated robot



- $q = (q_1, q_2, \ldots, q_{2n})$
- Number of dofs $= 2n$
- What is the topology?

An articulated object is a set of rigid bodies connected at the joints.

Example: protein backbone



- What are the possible representations?
- What is the number of dofs?
- What is the topology?

Configuration space

- Definitions and examples
- **Obstacles**
- Paths
- Metrics

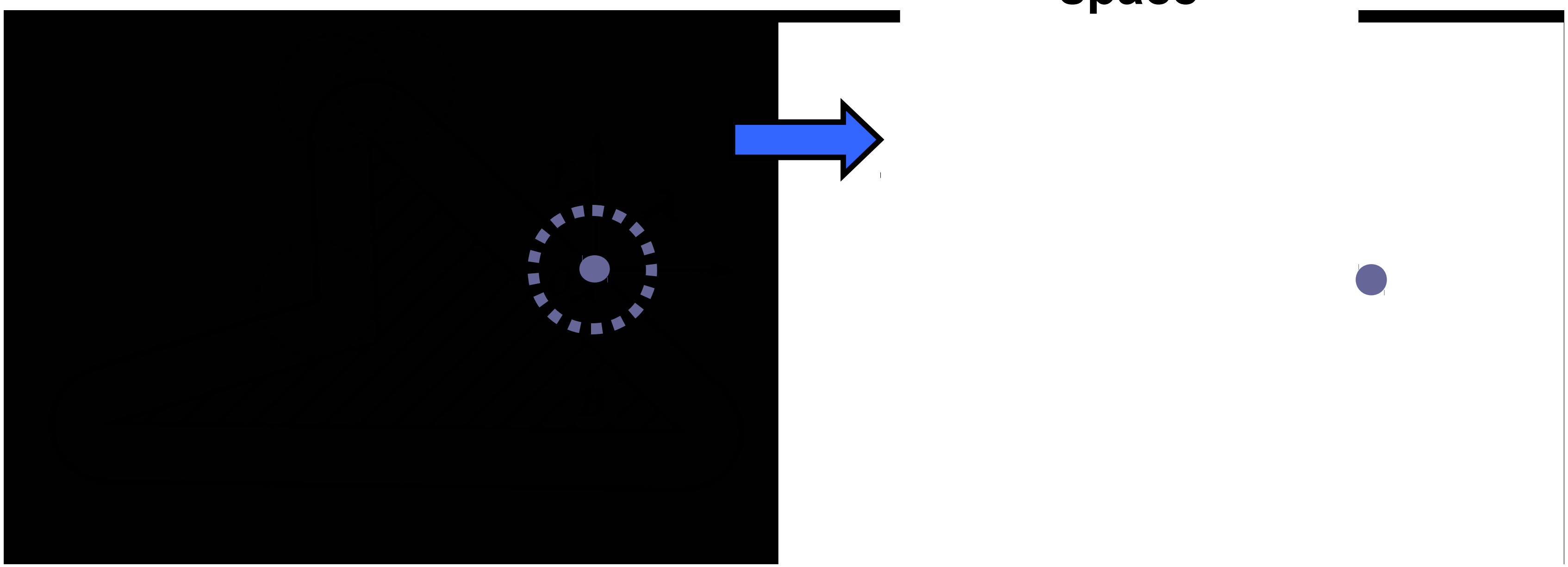
Obstacles in the configuration space

- A configuration q is collision-free, or **free**, if a moving object placed at q does not intersect any obstacles in the workspace.
- The **free space** F is the set of free configurations.
- A configuration space obstacle (**C-obstacle**) is the set of configurations where the moving object collides with workspace obstacles.

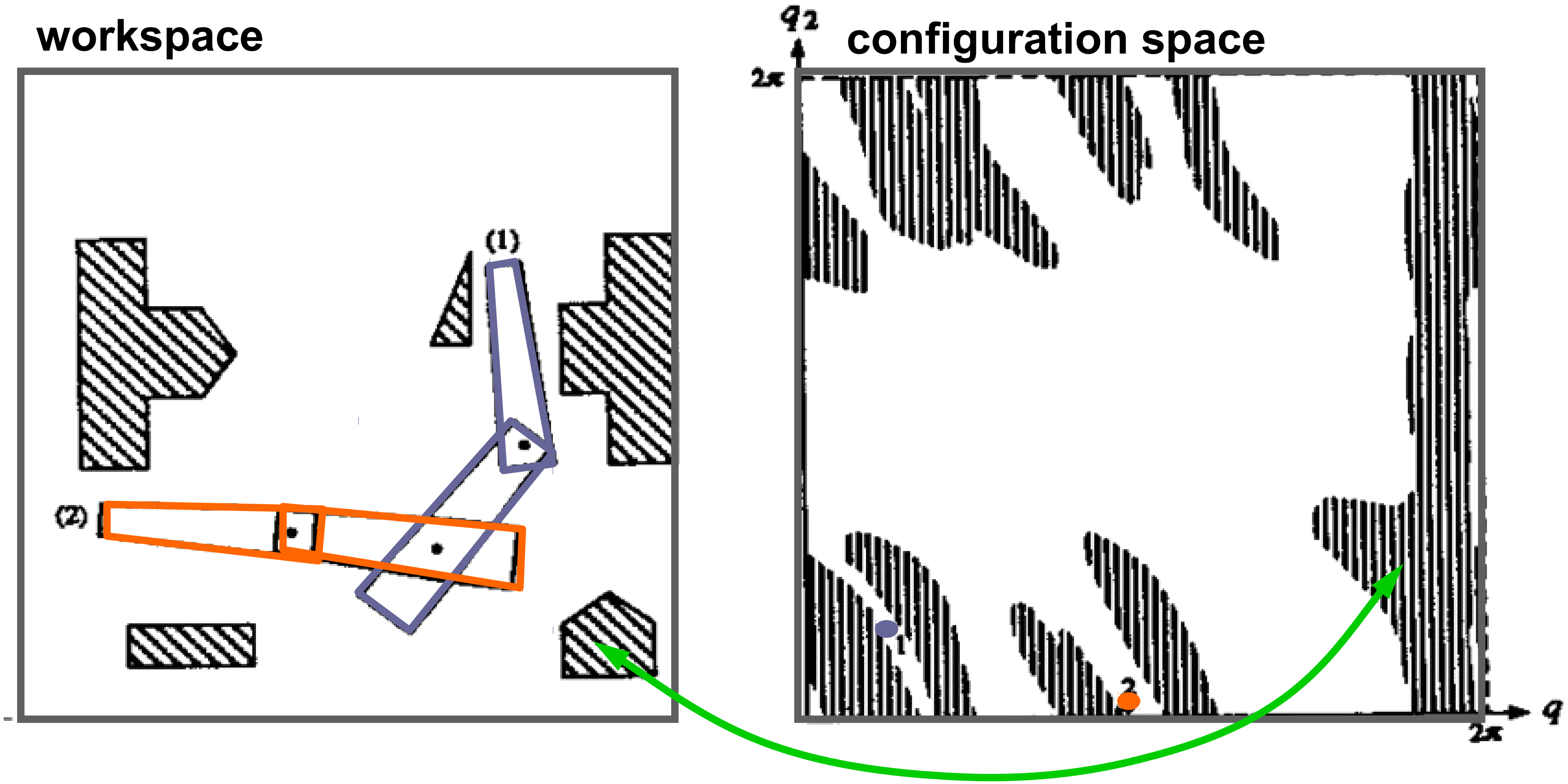
Disc in 2-D workspace

workspace
workspace

configuration space



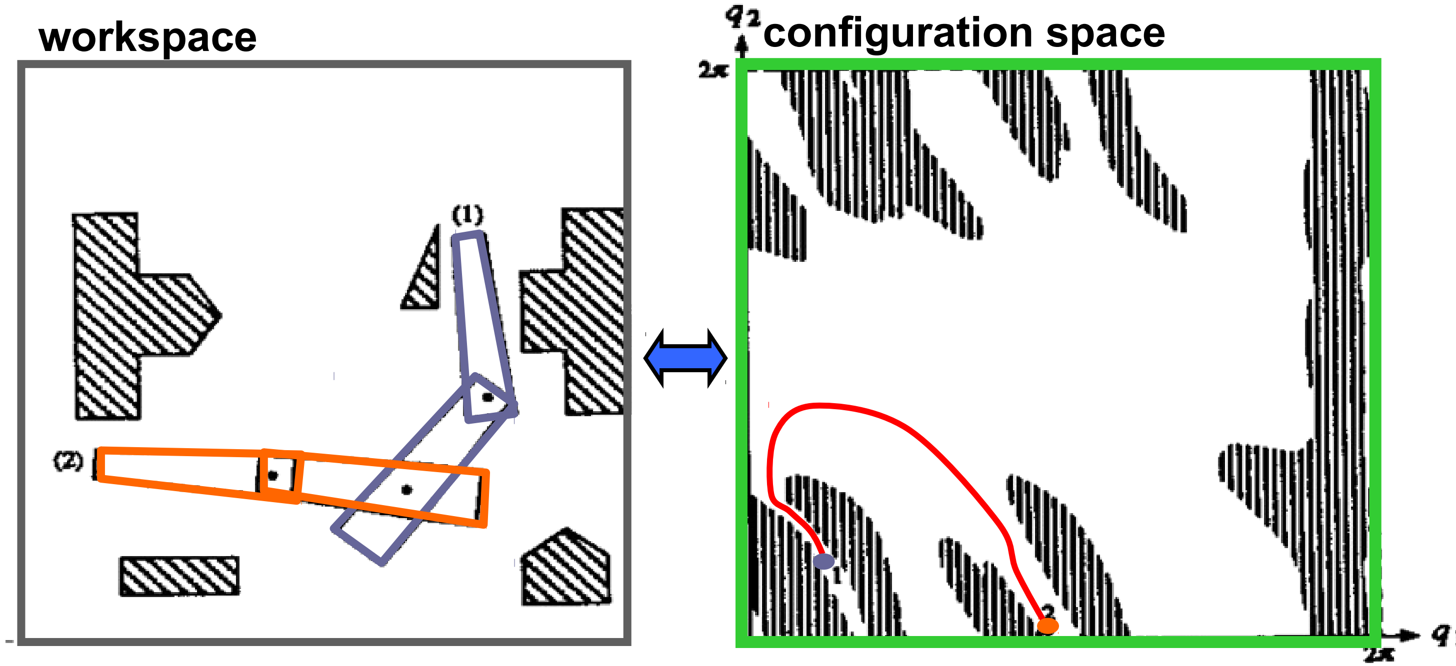
Articulated robot in 2-D workspace



Configuration space

- Definitions and examples
- Obstacles
- **Paths**
- Metrics

Paths in the configuration space



- A **path** in C is a continuous curve connecting two configurations q and q':

$$\tau : s \in [0,1] \rightarrow \tau(s) \in C$$

such that $\tau(0) = q$ and $\tau(1) = q'$.

Constraints on paths

- A **trajectory** is a path parameterized by time:

$$\tau : t \in [0, T] \rightarrow \tau(t) \in C$$

- Constraints
  - Finite length
  - Bounded curvature
  - Smoothness
  - Minimum length
  - Minimum time
  - Minimum energy
  - ...

Free space topology

- A **free** path lies entirely in the free space F.
- The moving object and the obstacles are modeled as closed subsets, meaning that they contain their boundaries.
- One can show that the C-obstacles are closed subsets of the configuration space C as well.
- Consequently, the free space F is an open subset of C. Hence, each free configuration is the center of a ball of non-zero radius entirely contained in F.

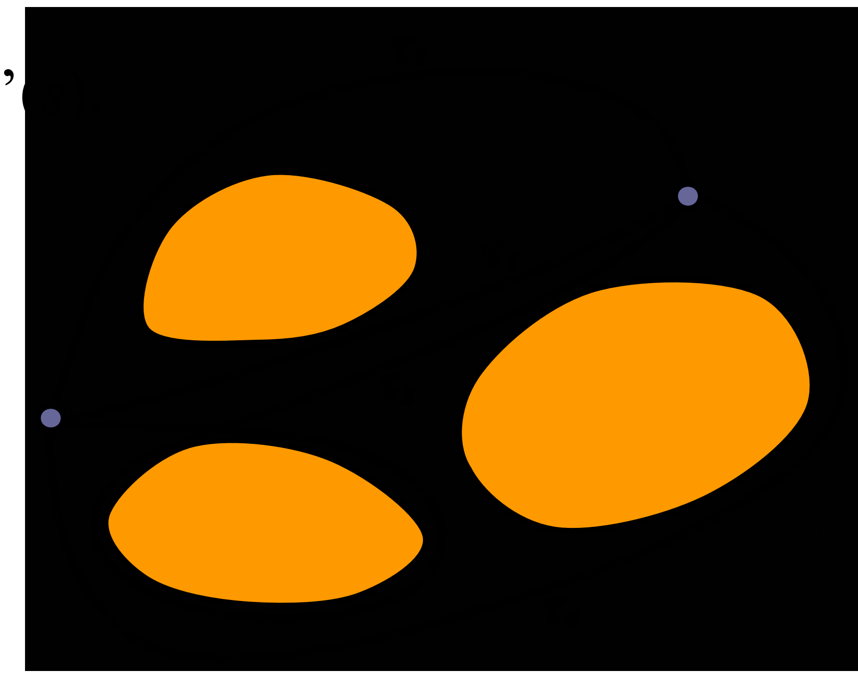
Homotopic paths

- Two paths τ and τ' with the same endpoints are **homotopic** if one can be continuously deformed into the other:

$$h:[0,1]\times[0,1]\rightarrow F$$

with $h(s,0) = \tau(s)$ and $h(s,1) = \tau'($

- A homotopic class of paths contains all paths that are homotopic to one another.



Connectedness of C-Space

- C is **connected** if every two configurations can be connected by a path.
- C is **simply-connected** if any two paths connecting the same endpoints are homotopic.
  Examples: R^2 or R^3
- Otherwise C is multiply-connected.
  Examples: S^1 and $SO(3)$ are multiply- connected:
  - In S^1, infinite number of homotopy classes
  - In $SO(3)$, only two homotopy classes

Configuration space

- Definitions and examples
- Obstacles
- Paths
- **Metrics**

Metric in configuration space

- A **metric** or **distance** function d in a configuration space C is a function

$$d : (q, q') \in C^2 \rightarrow d(q, q') \geq 0$$

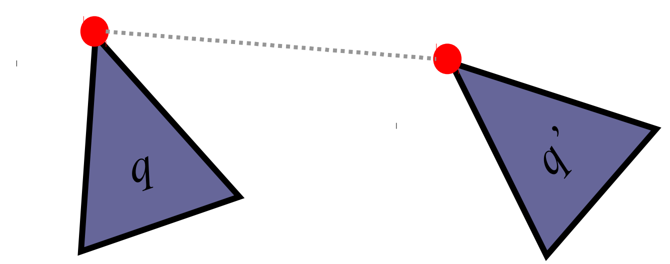
  such that
  - $d(q, q') = 0$ if and only if $q = q'$,
  - $d(q, q') = d(q', q)$,
  - $d(q, q') \leq d(q, q \backslash) + \mathrm{d} \backslash(\mathrm{q}, q')$.

Example

- Robot A and a point x on A
- $x(q)$: position of x in the workspace when A is at configuration q
- A distance d in C is defined by

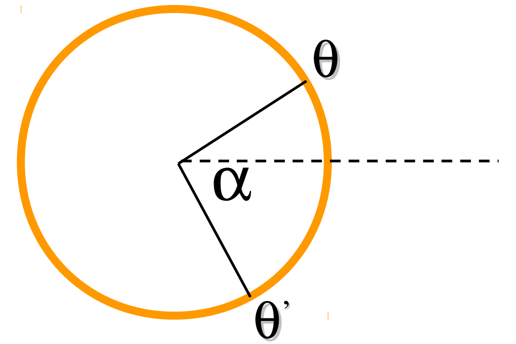
$$d(q, q') = \max_{x \in A} \| x(q) - x(q') \|$$

where $||x - y||$ denotes the Euclidean distance between points x and y in the workspace.



Examples in $R^2 \times S^1$

- Consider $R^2 \times S^1$
  - $q = (x, y, \theta)$, $q' = (x', y', \theta')$ with $\theta, \theta' \in [0,2\pi)$
  - $\alpha = \min \{ |\theta - \theta'| , 2\pi - |\theta - \theta'| \}$



- $d(q, q') = \text{sqrt}((x-x')^2 + (y-y')^2 + \alpha^2))$

- $d(q, q') = \text{sqrt}((x-x')^2 + (y-y')^2 + (\alpha r)^2)$, where r is the maximal distance between a point on the robot and the reference point

Summary on configuration space

- Parametrization
- Dimension (dofs)
- Topology
- Metric