Solving discrete time AQR problems by substituting into a standard LQR parameterization

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1 Problem

We are often interested in solving linear quadratic regulation (LQR) control problems for systems with affine dynamics (instead of linear) such as

$$x_{t+1} = Ax_t + Bu_t + c,$$

and quadratic cost functions with additive linear and constant terms such as

$$J(u_{1:T-1}) = x_T^T Q_f x^T + 2x^T q_f + \sum_{t=1}^{T-1} x_t^T Q x_t + 2x_t^T q + u_t^T R u_t + 2u_t^T r.$$

We refer to this as the affine quadratic regulation (AQR) problem. It turns out that it is possible to identify a set of equations for this system similar to the standard Riccati equation using a variation of the standard LQR derivation. However, since this the derivation and the resulting equations are confusing, this paper outlines a method for solving the AQR problem using standard LQR Riccati equations. This formulation is based on and is nearly the same as that in Pieter Abbeel's course notes for Robotics 2 at Stanford University.

2 Solution

The standard LQR formulation finds an optimal solution to the linear system,

$$x_{t+1} = Ax_t + Bu_t,$$

with a quadratic cost function,

$$J(u_{1:T-1}) = x_T^T Q_f x^T + \sum_{t=1}^{T-1} x_t^T Q x_t + u_t^T R u_t.$$

The key insight is that given an AQR problem, we can define the following matrices:

$$\bar{A} = \begin{pmatrix} A & c - BR^{-1}r \\ 0 & 1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B \\ 0 \end{pmatrix},$$

$$\bar{Q}_f = \begin{pmatrix} Q_f & q_f \\ q_f^T & \eta \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} Q & q \\ q^T & \eta \end{pmatrix},$$

where η is an arbitrary constant, and vectors:

$$\bar{x}_t = \begin{pmatrix} x_t \\ 1 \end{pmatrix}, \quad \bar{u}_t = u_t + R^{-1}r.$$

After working the math, one can see that:

$$\begin{pmatrix} Ax_t + Bu_t + c \\ 1 \end{pmatrix} = \bar{A}\bar{x}_t + \bar{B}\bar{u}_t,$$

and

$$x_T^T Q_f x^T + 2x^T q_f + \sum_{t=1}^{T-1} x_t^T Q x_t + 2x_t^T q + u_t^T R u_t + 2u_t^T r + \eta' = \bar{x}_T^T \bar{Q}_f \bar{x}^T + \sum_{t=1}^{T-1} \bar{x}_t^T \bar{Q} \bar{x}_t + \bar{u}_t^T \bar{R} \bar{u}_t,$$

where η' is an arbitrary constant. Therefore, if we construct $\bar{A}$, $\bar{B}$, $\bar{Q}$, *etc.* and solve using the standard DLQR equations, we will get a solution to the AQR problem.