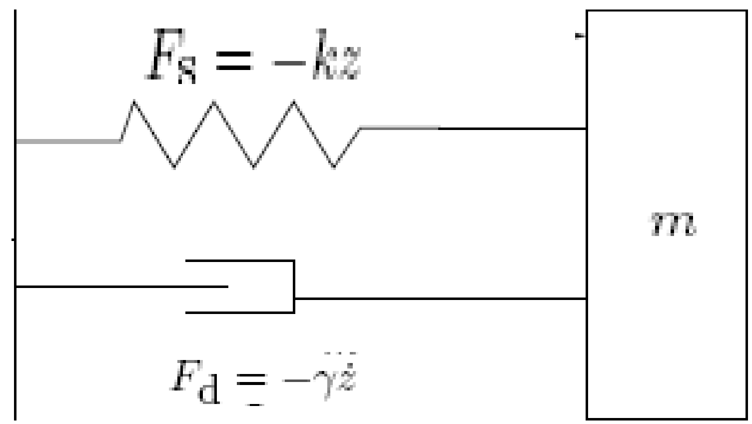
Robotic Motion Planning: Controls Primer

Robotics Institute 16-735
http://www.cs.cmu.edu/~motion

Howie Choset
http://www.cs.cmu.edu/~choset

State Space



$$m\ddot{z}(t) = -\gamma\dot{z}(t) - kz(t)$$

Model of mass spring damper system

z(t) position, *ż(t)* velocity
t_0 initial time, $z(t_0)$, $\dot{z}(t_0)$ initial position & velocity

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{m}\left(\gamma\dot{z}(t) + kz(t)\right) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{m}\left(\gamma x_2(t) + kx_1(t)\right) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t)$$

State space representation of mass spring damper system (1st order ODE)

$$\dot{x}(t) = Ax(t); \qquad x(t_0) = x_0 \qquad\qquad x(t) \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n\times n}$$

Vector Field

assigns a vector Ax to each point x in the state space. $\dot{x}(t) = Ax(t); \qquad x(t_0) = x_0$

$$x(t) = e^{A(t-t_0)}x_0$$

$$\begin{aligned} e^{A(t-t_0)} &= \sum_{i=0}^{\infty} \frac{A^i (t-t_0)^i}{i!} \\ &= I_{n\times n} + A(t-t_0) + \frac{A^2 (t-t_0)^2}{2!} + \cdots \end{aligned}$$

Stability

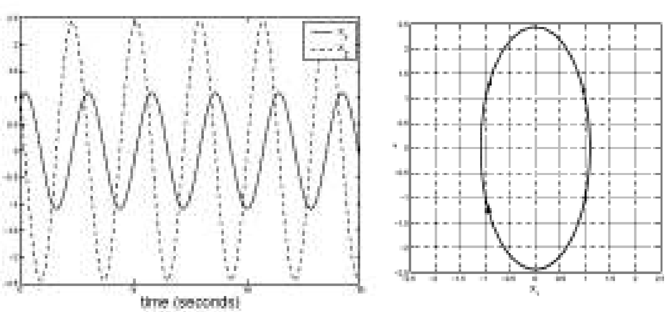
$$x(t) = e^{A(t-t_0)}x_0 \qquad \dot{x}(t) = Ax(t); \qquad x(t_0) = x_0$$

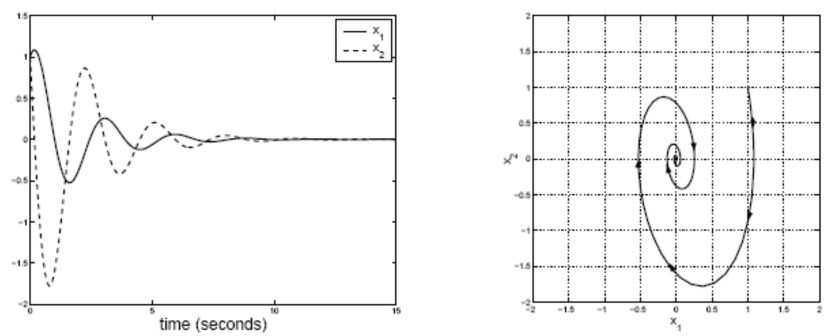
Equilibrium $\dot{x} = 0$ which occurs here at $x_e = 0$

for every $\varepsilon > 0$, there exists a δ for initial conditions $\|x_e - x(t_0)\| < \delta$

Stability:
$x(t)$ satisfies $\|x_e - x(t)\| < \epsilon$

Asymptotic Stability:
$\|x_e - x(t)\| \rightarrow 0$ as $t \rightarrow \infty$





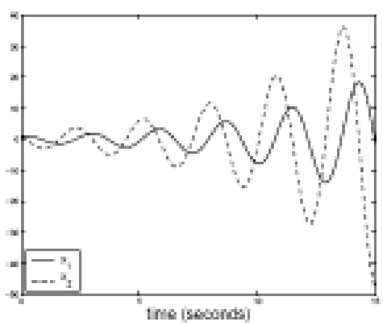
Unstable: Neither

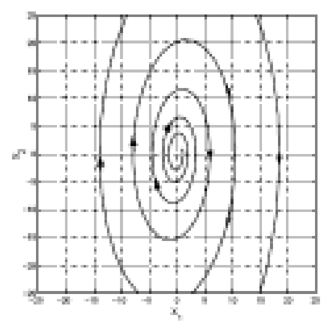
Stability and Eigenvalues

let $\lambda_i,\ i \in \{1, 2, \ldots, n\}$ *denote the eigenvalues of* A. *Let* $\mathrm{re}(\lambda_i)$ *denote the real part of* λ_i *Then the following holds:*

1. $x_e = 0$ *is stable if and only if* $\mathrm{re}(\lambda_i) \leq 0$ *for all* i.
2. $x_e = 0$ *is asymptotically stable if and only if* $\mathrm{re}(\lambda_i) < 0$ *for all* i.
3. $x_e = 0$ *is unstable if and only if* $\mathrm{re}(\lambda_i) > 0$ *for any* i.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) \qquad \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$





Negative damping….

Apply a force (a control input)

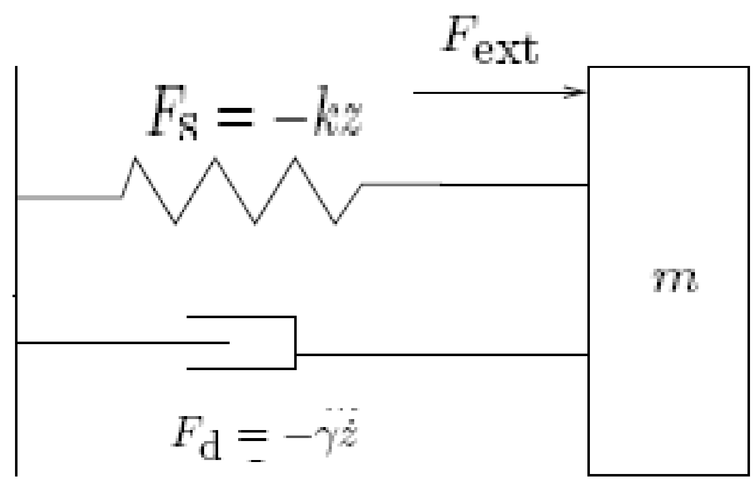
$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad x(t_0) = x_0.$$

$x(t) \in \mathbb{R}^n$

$u(t) \in \mathbb{R}^m$

$B \in \mathbb{R}^{n \times m}$



Controlability

For any initial condition $x(t_0)$, there exists an input $u(t)$ that drives the solution $x(t)$ to the origin*

$$\dot{x}(t) = Ax(t) + Bu(t); \qquad x(t_0) = x_0$$

The LTI is control system controllable if and only if W has rank n where

$$W = [\ B\ AB\ A^2B\ \dots\ A^{n-1}B]$$

*(assuming the origin is an equilibrium point for the unforced system)

Closed Loop Stability: Pole Placement

Make an unforced unstable system stable

$$\dot{x}(t) = Ax(t) + Bu(t)$$

State dependent control law

$$u(t) = -Kx(t) \quad K \in \mathbb{R}^{m \times n}$$

$$\dot{x}(t) = \boxed{(A - BK)}x(t)$$

look at the eigenvalues

For a real valued matrix, if an eigenvalue *a + ib* has *b ≠ 0*, then *a – ib* is an eigenvalue

So,

$$\bar{\Lambda} = \{\lambda_i | i \in \{1, 2, \ldots, n\}\}$$

Is ***allowable*** if for each λ_i that has an imaginary part, there is a λ_j that is a complex conjugate

<u>Assume</u>

This system is controllable, i.e., the pair (A, B) is controllable

B is full rank

$\bar{\Lambda} = \{\acute{\lambda}_i | i \in \{1, 2, \ldots, n\}\}$ is an allowable set of complex numbers

Then there exists an constant matrix $K \in \mathbb{R}^{m \times n}$ so that $(A - BK)$ *is equal to* Λ

What does this mean?

- To stabilize

$$\dot{x}(t) = Ax(t) + Bu(t)$$

State dependent control law

$$u(t) = -Kx(t) \quad K \in \mathbb{R}^{m \times n}$$

$$\dot{x}(t) = (A - BK)x(t)$$

- Find a *K* so that the Eigenvalues of *A* – *BK* have negative real values and are complex conjugates.

- The famous LQR does this by optimizing a user-defined cost function.

-

Example

- Mass-spring-damper (negative damping - why)

$$A - BK = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [k_1 \; k_2] = \begin{bmatrix} 0 & 1 \\ -\frac{k+k_1}{m} & \frac{\gamma+k_2}{m} \end{bmatrix}$$

- Eigenvalues

$$\frac{(-\gamma - k_2) \pm \sqrt{(-\gamma - k_2)^2 - 4(k - k_1)m}}{2m},$$

- Chose $k_2 \text{ such that } -\gamma - k_2 < 0$

- This is like adding positive damping

Observing LTI Systems

- Sadly, one cannot always sense all of the state variables

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \qquad x(t_0) = x_0, \\ y(t) &= Cx(t), \end{aligned}$$

state	*control*	*output*	
$x(t) \in \mathbb{R}^n$	$u(t) \in \mathbb{R}^m$	$y(t) \in \mathbb{R}^p$	$C \in \mathbb{R}^{p \times n}$.

May not be invertible (nor square)

- Example: can only sense or measure velocity

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t), \\ y(t) &= [0\ 1]\, x(t), \end{aligned}$$

- Can we recover the state from the observations??

Observable

*A system is **observable** if one can determine the initial state by observing the output and knowing the controls over some period of time*

The system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \qquad x(t_0) = x_0 \\ y(t) &= Cx(t), \end{aligned}$$

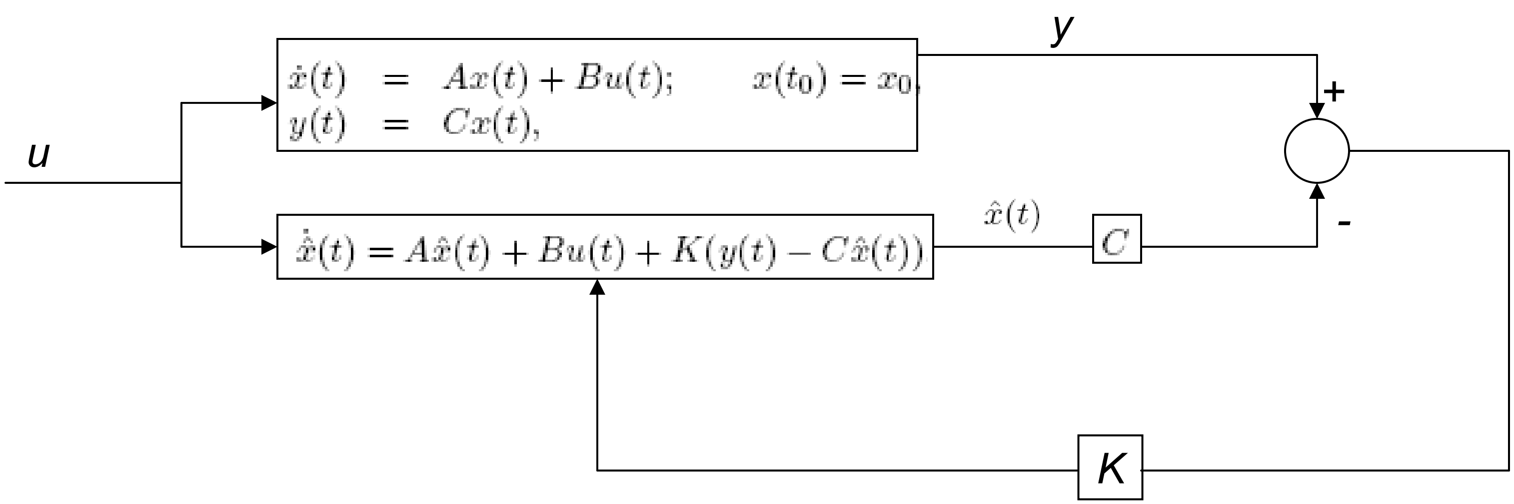
Is observable if $\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has rank n

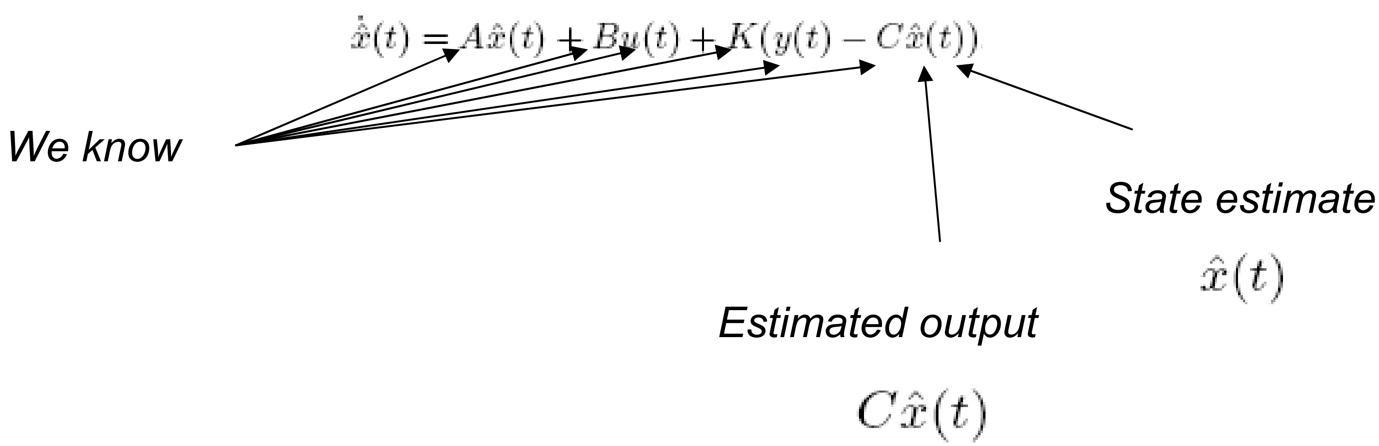
The pair (A,C) is considered observable.

(A,C) observable if and only if (A^T, C^T) is controllable

If (A,B) controllable and (A,C) observable, then (A,B,C) is ***minimal***

Observer





Copy of original system with a correcting term $K(y(t) - C\hat{x}(t))$ *which is the difference between the output and estimated output*

How does this work

- Consider the error $e(t) = x(t) - \hat{x}(t)$

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= Ax(t) + Bu(t) - (A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))) \\ &= A(x(t) - \hat{x}(t)) - K(Cx(t) - C\hat{x}(t)) \\ &= (A - KC)e(t) \end{aligned}$$

- As $e(t) \to 0, \text{ then } \hat{x}(t) \to x(t)$
- Chose *K* so that $\dot{e}(t) = (A - KC)e(t)$ asymp. Stable
- Look at eigenvalues of *A - KC* (not quite right form)
- Look at eigenvalues of $A^t - C^tK^t$ (same eigenvalues)
- Make sure such eigenvalues are allowable
  - Make sure (A^t, C^t) is controllable
  - C^t is full rank
- Same as saying *(A,C)* is observable
- So it is a matter of choosing *K*

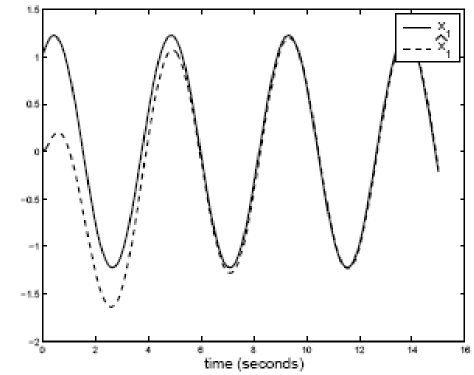
Example

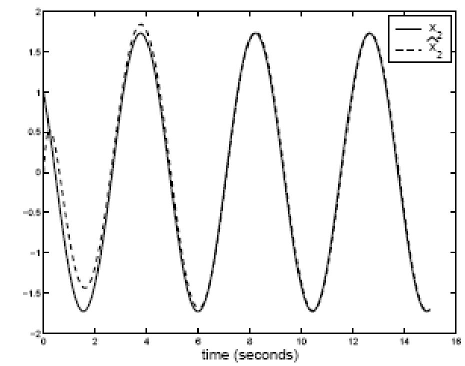
$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t),$$
$$y(t) = [0\ 1]\, x(t),$$

$$A - KC = \begin{bmatrix} 0 & 1 - k_1 \\ -\frac{k}{m} & -\frac{\gamma}{m} - k_2 \end{bmatrix}$$

$$\frac{-(\gamma + mk_2) \pm \sqrt{(-\gamma - mk_2)^2 - 4m(k - k_1)}}{2m}$$

Chose k_2 such that $-\gamma - mk_2 < 0$





$m = 1,\ k = 2,$ and $\gamma = 0,$

Chose $K = [0\ 2]^T$

Discrete Time

$$\dot{x}(t_0 + kT) \approx \frac{x(k+1) - x(k)}{T} \approx Ax(k) + Bu(k)$$

$$x(k+1) \approx x(k) + TAx(k) + TBu(k)$$

$F = I_{n \times n} + TA$, $G = TB$, and $H = C$;

$$\begin{aligned} \dot{x}(k+1) &= Fx(k) + Gu(k); \qquad x(0) = x_0 \\ y(k) &= Hx(k) \end{aligned}$$

Properties

- Stability of *x(k+1) = F(x) x(k)*
  - *Let* $\lambda_i,\ i \in \{1, 2, \ldots, n\}$ *denote the eigenvalues of F*

$x_e = 0$ *is stable if and only if* $|\lambda_i| \leq 1$ *for all* i.

$x_e = 0$ *is asymptotically stable if and only if* $|\lambda_i| < 1$ *for all* i

$x_e = 0$ *is unstable if and only if* $|\lambda_i| > 1$ *for any* i.

- Controllability of *(F,G)*

$$[G\ FG\ F^2G\ \cdots\ F^{n-1}G] \text{ has rank } n$$

- *Observability of (F,H) – if* F^t, H^t *is controllable*