Robotic Motion Planning: Controls Primer

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State Space

State space representation of mass spring damper system (1st order ODE)

\[
m\ddot{z}(t) = -\gamma \dot{z}(t) - kz(t)
\]

\(z(t)\) position, \(\dot{z}(t)\) velocity
\(t_0\) initial time, \(z(t_0), \dot{z}(t_0)\) initial position & velocity

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}
\]

\[
\dot{x}(t) = \begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{m}(\gamma \dot{z}(t) + kz(t)) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{1}{m}(\gamma x_2(t) + kx_1(t)) \end{bmatrix}
\]

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k/m & -\gamma/m \end{bmatrix} x(t)
\]

\[
\dot{x}(t) = Ax(t); \quad x(t_0) = x_0 \quad \text{for } x(t) \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n\times n}
\]
Vector Field

assigns a vector $Ax$ to each point $x$ in the state space.  \[ \dot{x}(t) = Ax(t); \quad x(t_0) = x_0 \]

\[ x(t) = e^{A(t-t_0)}x_0 \]

\[ e^{A(t-t_0)} = \sum_{i=0}^{\infty} \frac{A^i(t-t_0)^i}{i!} \]

\[ = I_{n \times n} + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \cdots \]
Stability

\[ x(t) = e^{A(t-t_0)}x_0 \quad \dot{x}(t) = Ax(t); \quad x(t_0) = x_0 \]

**Equilibrium** \( \dot{x} = 0 \) which occurs here at \( x_e = 0 \)

for every \( \varepsilon > 0 \), there exists a \( \delta \) for initial conditions \( \|x_e - x(t_0)\| < \delta \)

**Stability:**
\( x(t) \) satisfies \( \|x_e - x(t)\| < \varepsilon \)

**Asymptotic Stability:**
\( \|x_e - x(t)\| \to 0 \) as \( t \to \infty \)

**Unstable:** Neither
Stability and Eigenvalues

Let \( \lambda_i, \ i \in \{1, 2, \ldots, n\} \) denote the eigenvalues of \( A \). Let \( \text{re}(\lambda_i) \) denote the real part of \( \lambda_i \). Then the following holds:

1. \( x_e = 0 \) is stable if and only if \( \text{re}(\lambda_i) \leq 0 \) for all \( i \).
2. \( x_e = 0 \) is asymptotically stable if and only if \( \text{re}(\lambda_i) < 0 \) for all \( i \).
3. \( x_e = 0 \) is unstable if and only if \( \text{re}(\lambda_i) > 0 \) for any \( i \).

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} -\gamma \pm \sqrt{\gamma^2 - 4km} \\ 2m \end{bmatrix}.
\]

Negative damping....
Apply a force (a control input)

\[
\dot{x}(t) = \begin{bmatrix} \frac{k}{m} & \frac{1}{m} \\ \frac{-k}{m} & \frac{-\gamma}{m} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t)
\]

\[
\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_0) = x_0,
\]

\[x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad B \in \mathbb{R}^{n \times m}\]
Controlability

For any initial condition $x(t_0)$, there exists an input $u(t)$ that drives the solution $x(t)$ to the origin* 

$$\dot{x}(t) = Ax(t) + Bu(t); \quad x(t_0) = x_0$$

The LTI is control system controllable if and only if $W$ has rank $n$ where

$$W = [ B \ AB \ A^2B \ ... \ A^{n-1}B]$$

*(assuming the origin is an equilibrium point for the unforced system)
Closed Loop Stability: Pole Placement

Make an unforced unstable system stable

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

State dependent control law

\[ \dot{x}(t) = (A - BK)x(t) \]

\[ u(t) = -Kx(t) \quad K \in \mathbb{R}^{m \times n} \]

For a real valued matrix, if an eigenvalue \( a + ib \) has \( b \neq 0 \), then \( a - ib \) is an eigenvalue

So,

\[ \Lambda = \{ \lambda_i | i \in \{1, 2, \ldots, n\} \} \]

Is allowable if for each \( \lambda_i \) that has an imaginary part, there is a \( \lambda_j \) that is a complex conjugate

Assume

This system is controllable, i.e., the pair \((A, B)\) is controllable

\( B \) is full rank

\[ \Lambda = \{ \lambda_i | i \in \{1, 2, \ldots, n\} \} \] is an allowable set of complex numbers

Then there exists an constant matrix \( K \in \mathbb{R}^{m \times n} \) so that \( (A - BK) \) is equal to \( \Lambda \)
What does this mean?

• To stabilize

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
\end{align*}
\]

\[
\dot{x}(t) = (A - BK)x(t) \\
\begin{align*}
\quad u(t) &= -Kx(t) & K \in \mathbb{R}^{m \times n} \\
\end{align*}
\]

State dependent control law

• Find a $K$ so that the Eigenvalues of $A - BK$ have negative real values and are complex conjugates.

• The famous LQR does this by optimizing a user-defined cost function.
Example

• Mass-spring-damper (negative damping - why)

\[ A - BK = \begin{bmatrix} \frac{0}{m} & \frac{1}{m} \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix} - \begin{bmatrix} \frac{0}{m} \\ \frac{1}{m} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} \frac{0}{m} & \frac{1}{m} \\ -\frac{k+k_1}{m} & \frac{\gamma+k_2}{m} \end{bmatrix} \]

• Eigenvalues

\[ (-\gamma - k_2) \pm \sqrt{(-\gamma - k_2)^2 - 4(k - k_1)m} \]

\[ \frac{2m}{2m} \]

• Chose \[ k_2 \text{ such that } -\gamma - k_2 < 0 \]

• This is like adding positive damping
Observing LTI Systems

• Sadly, one cannot always sense all of the state variables

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t); \quad x(t_0) = x_0, \\
y(t) &= Cx(t),
\end{align*}
\]

- state \( x(t) \in \mathbb{R}^n \)
- control \( u(t) \in \mathbb{R}^m \)
- output \( y(t) \in \mathbb{R}^p \)

- \( C \in \mathbb{R}^{p \times n} \)

May not be invertible (nor square)

• Example: can only sense or measure velocity

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix}
0 & 0 \\
-k & -1/m
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
1/m
\end{bmatrix} F(t), \\
y(t) &= \begin{bmatrix}
0 & 1
\end{bmatrix} x(t),
\end{align*}
\]

• Can we recover the state from the observations??
A system is **observable** if one can determine the initial state by observing the output and knowing the controls over some period of time.

The system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t); \quad x(t_0) = x_0 \\
y(t) &= Cx(t),
\end{align*}
\]

Is observable if

\[
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

has rank \(n\)

*The pair \((A,C)\) is considered observable.*

\((A,C)\) observable if and only if \((A^T, C^T)\) is controllable

If \((A,B)\) controllable and \((A,C)\) observable, then \((A,B,C)\) is **minimal**
We know

\[ \begin{aligned}
    \dot{x}(t) &= Ax(t) + Bu(t); \\
    y(t) &= Cx(t),
\end{aligned} \]

\[ \begin{aligned}
    \dot{x}(t) &= A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))
\end{aligned} \]

Observer

State estimate

Estimated output

\[ C\hat{x}(t) \]

Copy of original system with a correcting term \( K(y(t) - C\hat{x}(t)) \) which is the difference between the output and estimated output
How does this work

• Consider the error $e(t) = x(t) - \hat{x}(t)$

\[
\begin{align*}
\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= Ax(t) + Bu(t) - (A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))) \\
&= A(x(t) - \hat{x}(t)) - K(Cx(t) - C\hat{x}(t)) \\
&= (A - KC)e(t)
\end{align*}
\]

• As $e(t) \to 0$, then $\hat{x}(t) \to x(t)$
• Chose $K$ so that $\dot{e}(t) = (A - KC)e(t)$ asymp. Stable
• Look at eigenvalues of $A - KC$ (not quite right form)
• Look at eigenvalues of $A^t - C^tK^t$ (same eigenvalues)
• Make sure such eigenvalues are allowable
  – Make sure $(A^t, C^t)$ is controllable
  – $C^t$ is full rank
• Same as saying $(A, C)$ is observable
• So it is a matter of choosing $K$
Example

\[
\dot{x}(t) = \begin{bmatrix} -\frac{k}{m} & -\frac{\gamma}{m} \\ -\frac{k}{m} & -\frac{\gamma}{m} - k_2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t),
\]

\[
y(t) = [0 \ 1] x(t),
\]

\[
A - KC = \begin{bmatrix} 0 & 1 - k_1 \\ -\frac{k}{m} & -\frac{\gamma}{m} - k_2 \end{bmatrix}
\]

\[
-(\gamma + mk_2) \pm \sqrt{(-\gamma - mk_2)^2 - 4m(k - k_1)}
\]

\[
2m
\]

Chose \( k_2 \) such that \(-\gamma - mk_2 < 0\)

\[
m = 1, \ k = 2, \text{ and } \gamma = 0.
\]

Chose \( K = [0 \ 2]^T \)
\[
\dot{x}(t_0 + kT) \approx \frac{x(k + 1) - x(k)}{T} \approx Ax(k) + Bu(k)
\]

\[
x(k + 1) \approx x(k) + TAx(k) + TBu(k)
\]

\[
F = I_{n \times n} + TA, \ G = TB, \ \text{and} \ H = C.
\]

\[
\begin{align*}
\dot{x}(k + 1) &= Fx(k) + Gu(k); \quad x(0) = x_0 \\
y(k) &= Hx(k)
\end{align*}
\]
Properties

- Stability of $x(k+1) = F(x) \cdot x(k)$
  - Let $\lambda_i, i \in \{1, 2, \ldots, n\}$ denote the eigenvalues of $F$

  $x_e = 0$ is stable if and only if $|\lambda_i| \leq 1$ for all $i$
  $x_e = 0$ is asymptotically stable if and only if $|\lambda_i| < 1$ for all $i$
  $x_e = 0$ is unstable if and only if $|\lambda_i| > 1$ for any $i$.

- Controllability of $(F,G)$

  \[
  [G \ F \ \dot{} \ \ F^2G \ \dot{} \ \ F^{n-1}G] \ \text{has rank} \ n
  \]

- Observability of $(F,H)$ – if $F^t$, $H^t$ is controllable