Last lecture

- Multiple-query PRM
- Lazy PRM (single-query PRM)
Single-Query PRM
Randomized expansion

Overview

1. Grow two trees from *Init* position and *Goal* configurations.

2. Randomly sample nodes around existing nodes.

3. Connect a node in the tree rooted at *Init* to a node in the tree rooted at the *Goal*.

![Expansion + Connection](image)
1. Pick a node $x$ with probability $1/w(x)$.

2. Randomly sample $k$ points around $x$.

3. For each sample $y$, calculate $w(y)$, which gives probability $1/w(y)$.
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$1/w(y_1) = 1/5$
Expansion

1. Pick a node $x$ with probability $1/w(x)$.

2. Randomly sample $k$ points around $x$.

3. For each sample $y$, calculate $w(y)$, which gives probability $1/w(y)$.

\[
\frac{1}{w(y_2)} = \frac{1}{2}
\]
1. Pick a node $x$ with probability $1/w(x)$.
2. Randomly sample $k$ points around $x$.
3. For each sample $y$, calculate $w(y)$, which gives probability $1/w(y)$.
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3. For each sample $y$, calculate $w(y)$, which gives probability $1/w(y)$. If $y$
   (a) has higher probability; (b) collision free; (c) can sees $x$
   then add $y$ into the tree.
Sampling distribution

- Weight $w(x) = \text{no. of neighbors}$
- Roughly $\Pr(x) \sim 1 / w(x)$
Effect of weighting

unweighted sampling

weighted sampling
Connection

- If a pair of nodes (i.e., $x$ in Init tree and $y$ in Goal tree) and $\text{distance}(x, y) < L$, check if $x$ can see $y$
  - YES, then connect $x$ and $y$
The program iterates between **Expansion** and **Connection**, until

- two trees are connected, or
- max number of expansion & connection steps is reached
Computed example
Expansive Spaces

Analysis of Probabilistic Roadmaps
Issues of probabilistic roadmaps

- Coverage
- Connectivity
Is the coverage adequate?

- It means that milestones are distributed such that almost any point of the configuration space can be connected by a straight line segment to one milestone.

![Bad Coverage](image1.png)  ![Good Coverage](image2.png)
Connectivity

- There should be a one-to-one correspondence between the connected components of the roadmap and those of $F$. 

![Bad](image1.png) ![Good](image2.png)
Narrow passages

- Connectivity is difficult to capture when there are narrow passages.
- Narrow passages are difficult to define.

Characterize coverage & connectivity? → Expansiveness
Definition: visibility set

Visibility set of $q$

- All configurations in $F$ that can be connected to $q$ by a straight-line path in $F$
- All configurations seen by $q$
**Definition: $\epsilon$-good**

- Every free configuration sees at least $\epsilon$ fraction of the free space, $\epsilon$ in $(0, 1]$. 

![Diagram showing 0.5-good and 1-good configurations](image)
Definition: lookout of a subset $S$

- Subset of points in $S$ that can see at least $\beta$ fraction of $F \setminus S$, $\beta$ is in $(0, 1]$.

0.4-lookout of $S$

This area is about 40% of $F \setminus S$

0.3-lookout of $S$
Definition: \((\varepsilon, \alpha, \beta)\)-expansive

- The free space \(F\) is \((\varepsilon, \alpha, \beta)\)-expansive if
  - Free space \(F\) is \(\varepsilon\)-good
  - For each subset \(S\) of \(F\), its \(\beta\)-lookout is at least \(\alpha\) fraction of \(S\). \(\varepsilon, \alpha, \beta\) are in \((0, 1]\)

\[ F \text{ is } \varepsilon\text{-good} \Rightarrow \varepsilon = 0.5 \]
\[ \beta\text{-lookout} \Rightarrow \beta = 0.4 \]
\[ \frac{\text{Volume(\(\beta\)-lookout)}}{\text{Volume(S)}} \Rightarrow \alpha = 0.2 \]

\(F\) is \((\varepsilon, \alpha, \beta)\)-expansive, where \(\varepsilon = 0.5, \alpha = 0.2, \beta = 0.4\).
Why expansiveness?

- $\epsilon, \alpha, \text{ and } \beta$ measure the expansiveness of a free space.

- Bigger $\epsilon$, $\alpha$, and $\beta \Rightarrow$ lower cost of constructing a roadmap with good connectivity and coverage.
Uniform sampling

- All-pairs path planning

- **Theorem 1**: A roadmap of uniformly-sampled milestones has the correct connectivity with probability at least

\[
\frac{16 \ln(1/\gamma)}{\varepsilon \alpha} + \frac{6}{\beta} \quad 1 - \gamma.
\]
**Definition: Linking sequence**

$p_{n+1}$ is chosen from the lookout of the subset seen by $p, p_1, \ldots, p_n$.
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$p_{n+1}$ is chosen from the lookout of the subset seen by $p$, $p_1, \ldots, p_n$. 
Space occupied by linking sequences
Size of lookout set

A C-space with larger lookout set has **higher probability** of constructing a linking sequence.
Lemmas

- In an expansive space with large $\varepsilon, \alpha, \text{ and } \beta$, we can obtain a linking sequence that covers a large fraction of the free space, with high probability.
Theorem 1

- Probability of achieving good connectivity increases exponentially with the number of milestones (in an expansive space).

- If $(\epsilon, \alpha, \beta)$ decreases $\rightarrow$ then need to increase the number of milestones (to maintain good connectivity)
Theorem 2

- Probability of achieving good coverage, **increases exponentially** with the number of milestones (in an expansive space).
Probabilistic completeness

In an expansive space, the probability that a PRM planner fails to find a path when one exists goes to 0 exponentially in the number of milestones (≈ running time).

[Hsu, Latombe, Motwani, 97]
Summary

- Main result
  - If a C-space is expansive, then a roadmap can be constructed efficiently with good connectivity and coverage.

- Limitation in practice
  - It does not tell you when to stop growing the roadmap.
  - A planner stops when either a path is found or max steps are reached.
Extensions

- Accelerate the planner by automatically generating intermediate configurations to decompose the free space into expansive components.
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- Use geometric transformations to increase the expansiveness of a free space, e.g., widening narrow passages.
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- Integrate the new planner with other planner for multiple-query path planning problems.

Questions?
Two tenets of PRM planning

- A relatively small number of milestones and local paths are sufficient to capture the connectivity of the free space.
  - Exponential convergence in expansive free space (probabilistic completeness)

- Checking sampled configurations and connections between samples for collision can be done efficiently.
  - Hierarchical collision checking