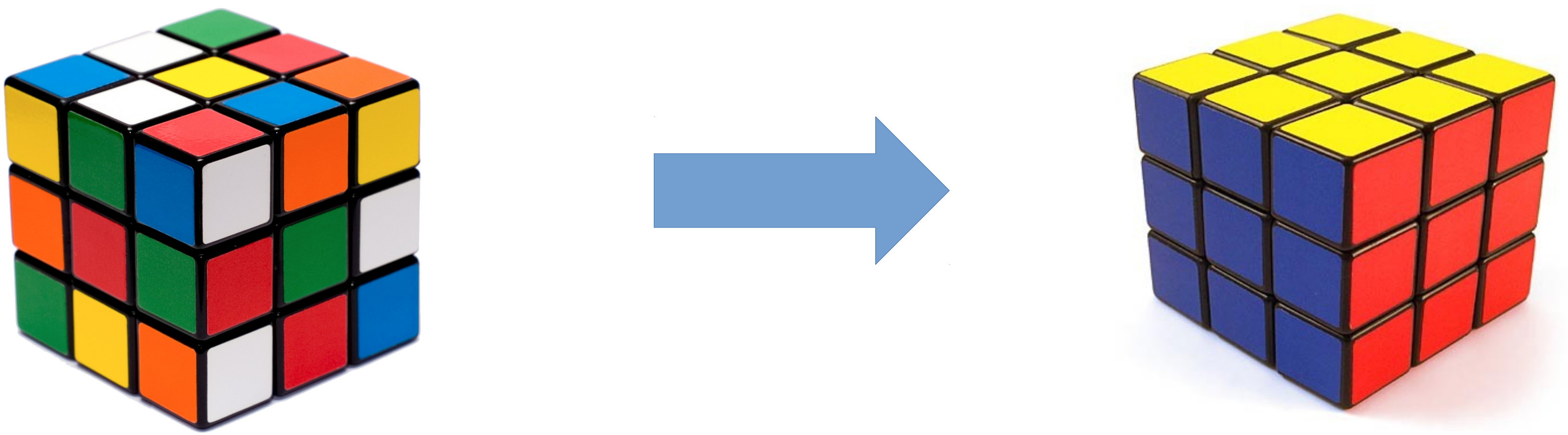
Graph Search

Rob Platt
Northeastern University

Some images and slides are used from:
AIMA
CS188 UC Berkeley

What is graph search?



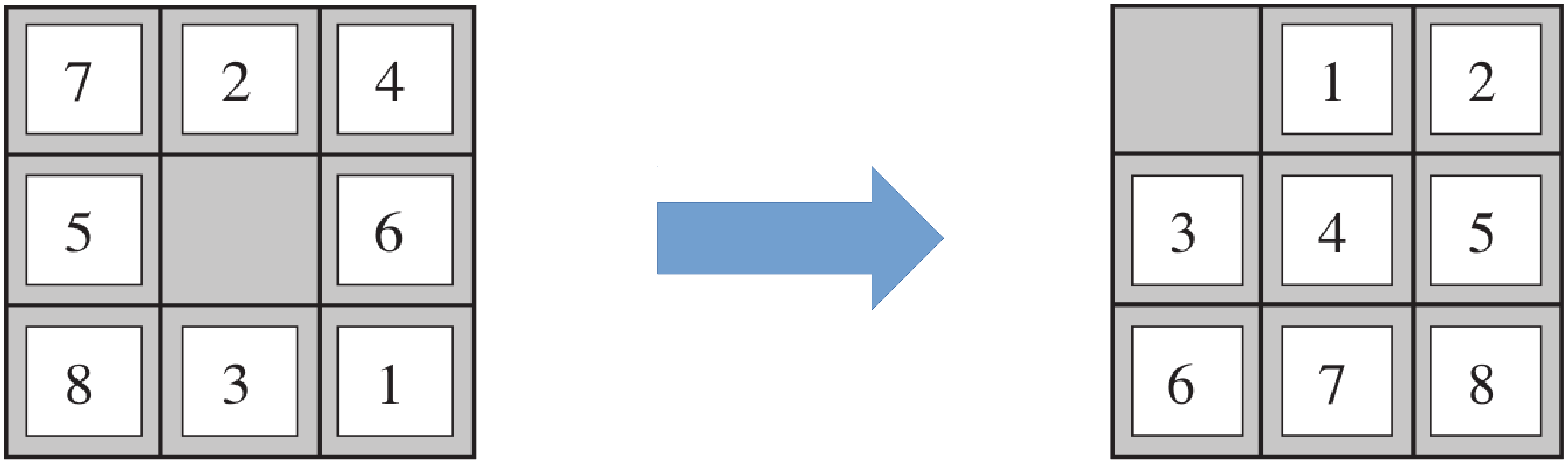
Start state

Goal state

Graph search: find a path from start to goal

- what are the states?
- what are the actions (transitions)?
- how is this a graph?

What is graph search?



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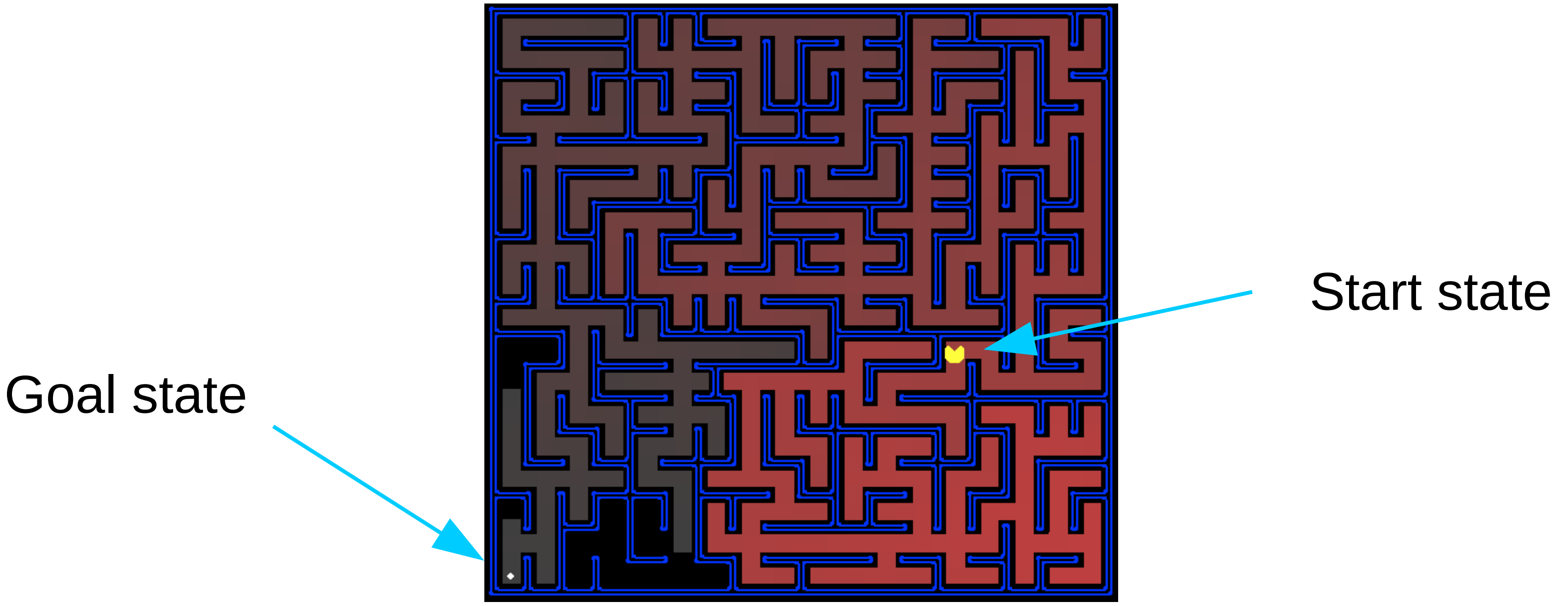

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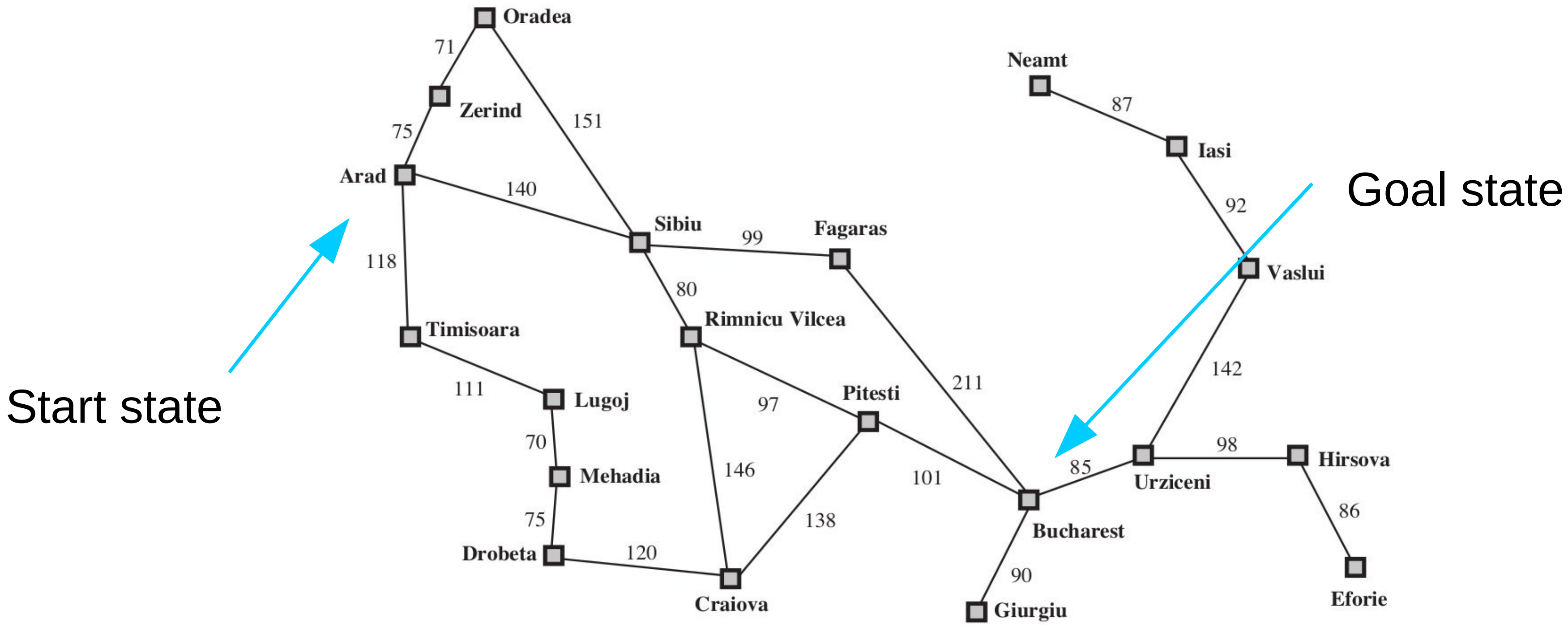


Graph search: find a path from start to goal

– what are the states?

– what are the actions (transitions)?

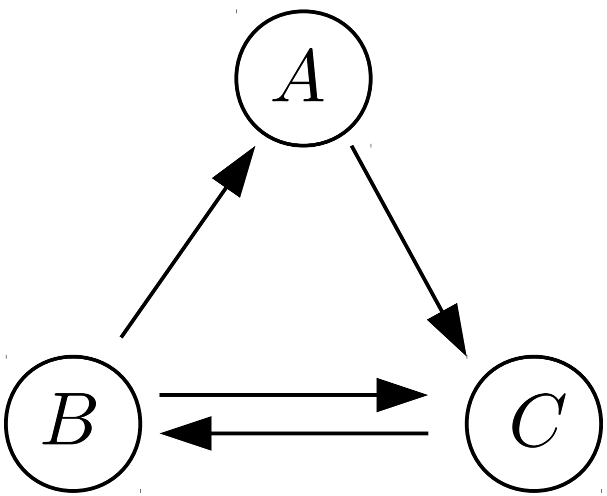
– how is this a graph?

What is a graph?

Graph: $G = (V, E)$

Vertices: V

Edges: E



Directed graph

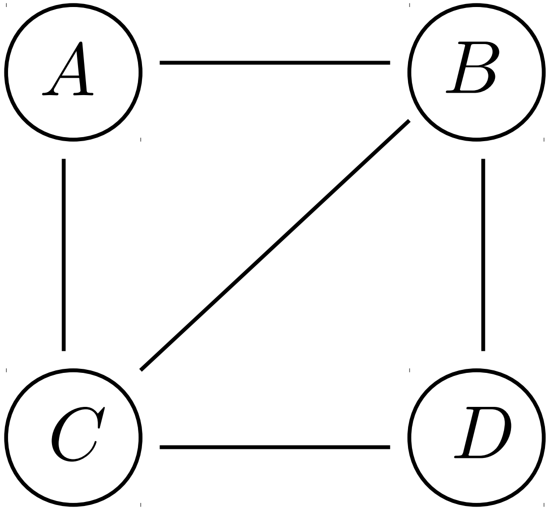
$$V = \{A, B, C\}$$
$$E = \{(B, A), (A, C), (B, C), (C, B)\}$$

What is a graph?

Graph: $G = (V, E)$

Vertices: V

Edges: E



<u>Undirected</u> graph

$$V = \{A, B, C, D\}$$

$$E = \{\{A, C\}, \{A, B\}, \{C, D\}, \{B, D\}, \{C, B\}\}$$

What is a graph?

Graph: $G = (V, E)$

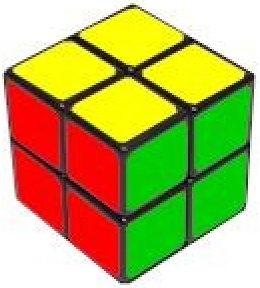
Vertices: $V \longleftarrow$ Also called *states*

Edges: $E \longleftarrow$ Also called *transitions*

Defining a graph: example

$V = ?$

$E = ?$



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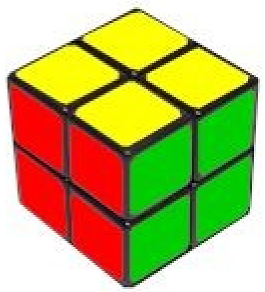
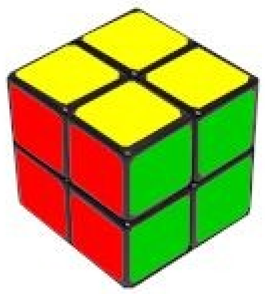


$V = ?$ ← How many states?

$E = ?$

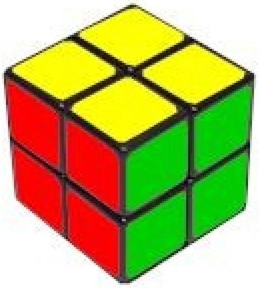
Defining a graph: example



$V = ?$ $\longleftarrow$ $|V| = 8! \times 3^8$

$E = ?$

Defining a graph: example



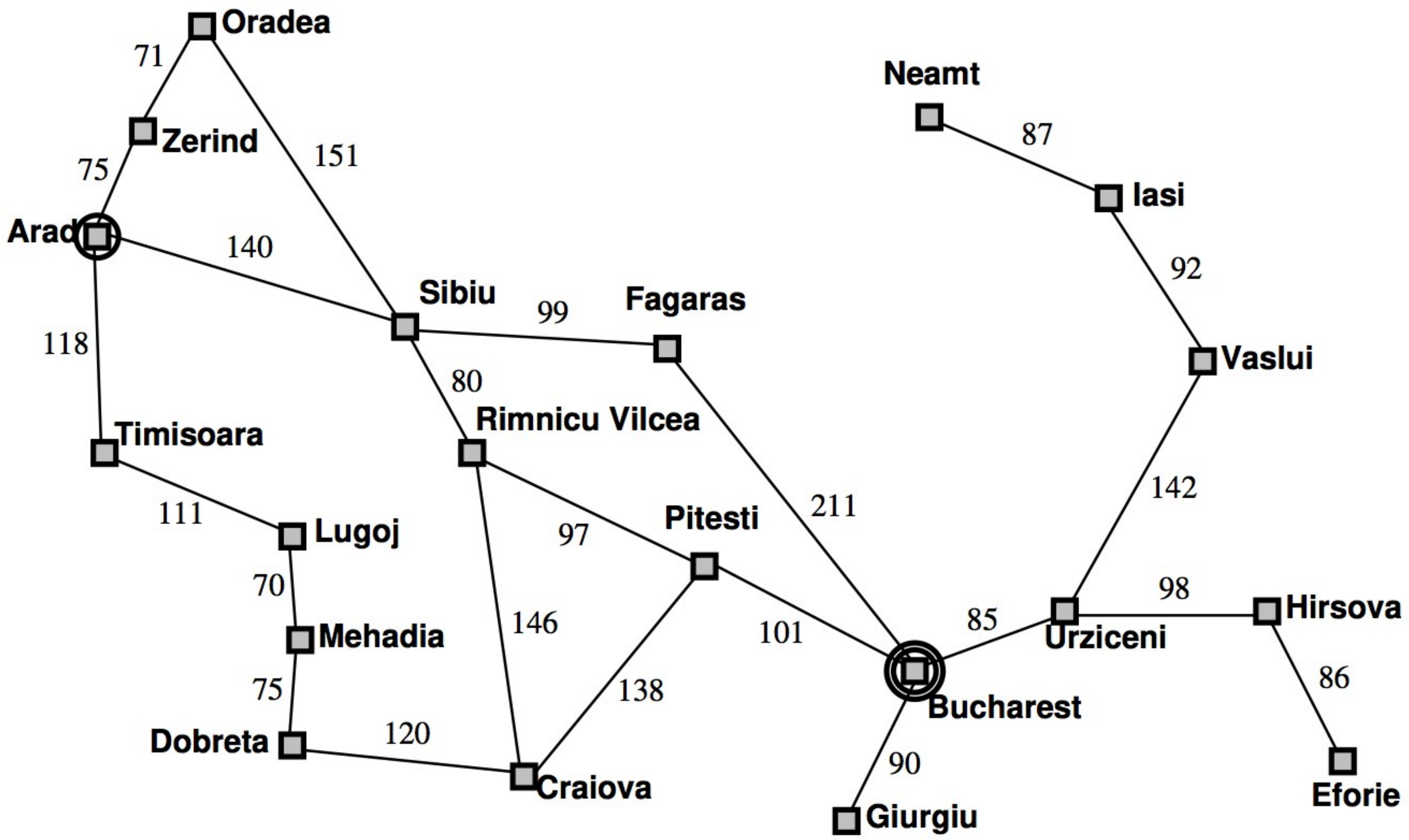
$V = ?$

$E = ?$ ← Pairs of states that are “connected” by one turn of the cube.

Example: Romania

- On holiday in Romania; currently in Arad. Flight leaves tomorrow from Bucharest
- Formulate goal: Be in Bucharest
- Formulate problem:
  - states: various cities
  - actions: drive between cities
- Find solution:
  - sequence of cities, e.g., Arad, Sibiu, Fagaras, Bucharest

Graph search



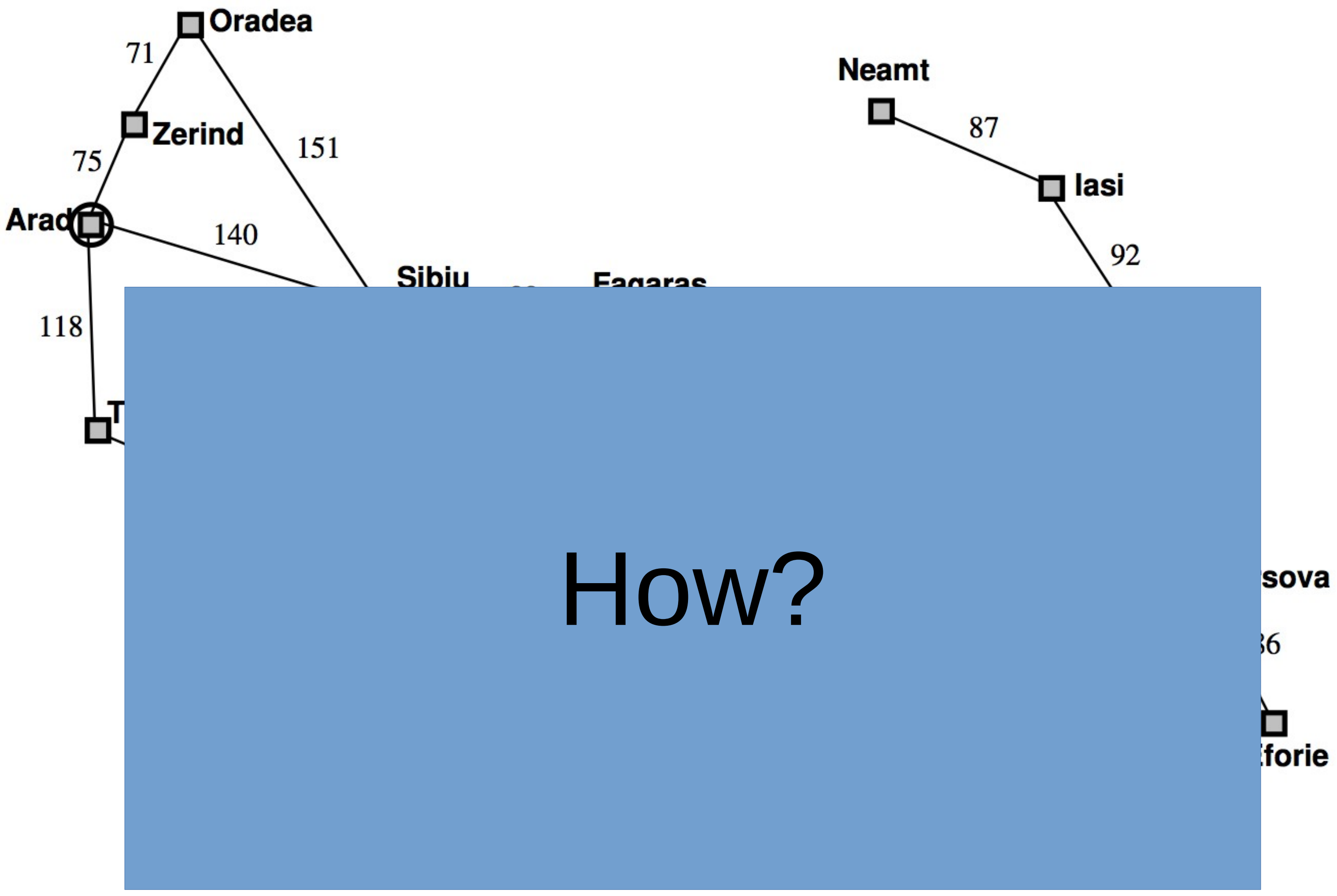
<u>Given</u>: a graph, *G*

<u>Problem</u>: find a path from A to B

– A: start state

– B: goal state

Graph search



Problem: find a path from A to B

– A: start state

– B: goal state

Problem formulation

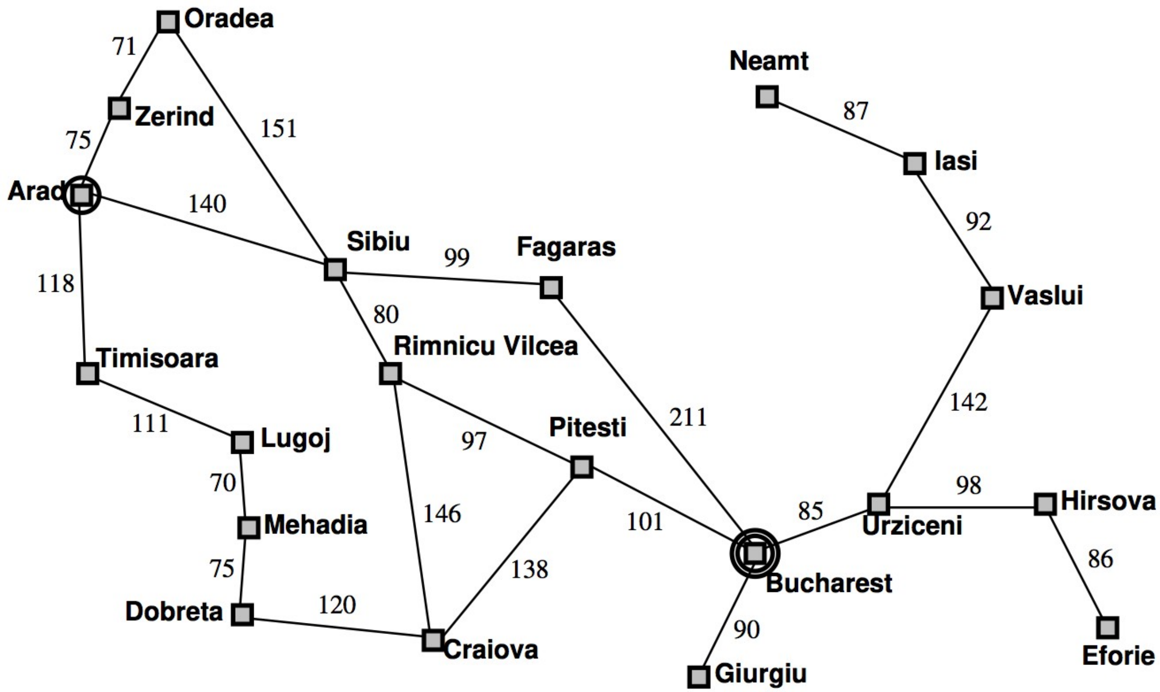
A problem is defined by four items:

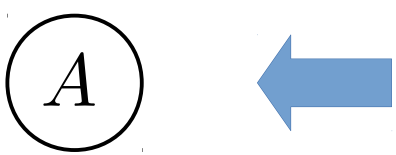
- initial state e.g., “at Arad”
- successor function $S(x)$ = set of action–state pairs

  e.g., $S(\text{Arad}) = \{\langle \text{Arad} \rightarrow \text{Zerind}, \text{Zerind}\rangle, \ldots\}$
- goal test, can be explicit, e.g., x = “at Bucharest” implicit, e.g., $NoDirt(x)$
- path cost (additive)

  e.g., sum of distances, number of actions executed, etc. $c(x, a, y)$ is the step cost, assumed to be ≥ 0
- A solution is a sequence of actions leading from the initial state to a goal state

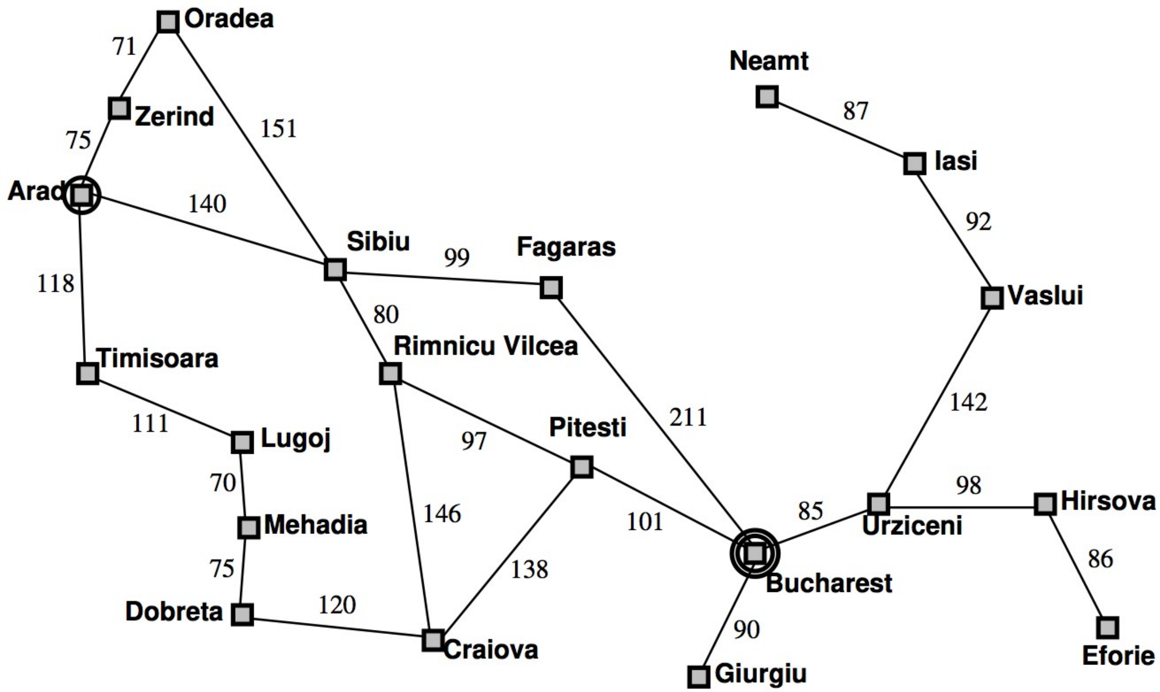
A search tree

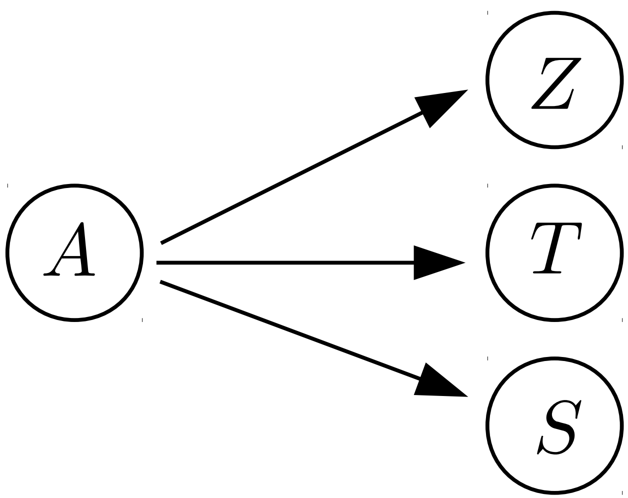




Start at A

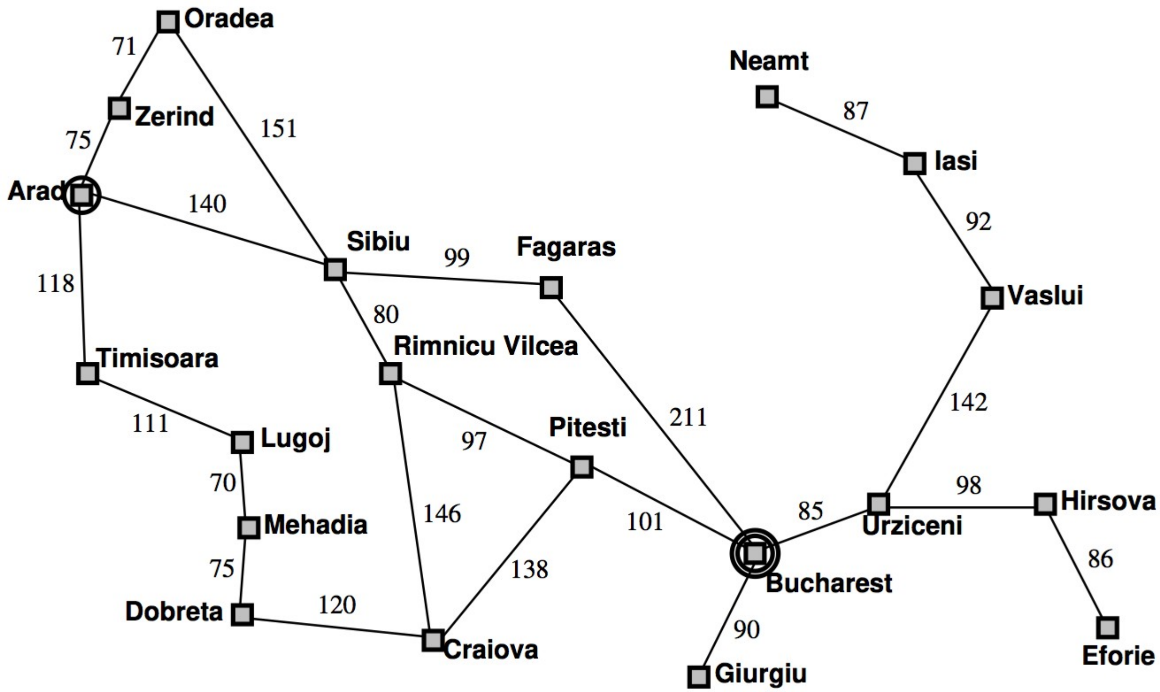
A search tree

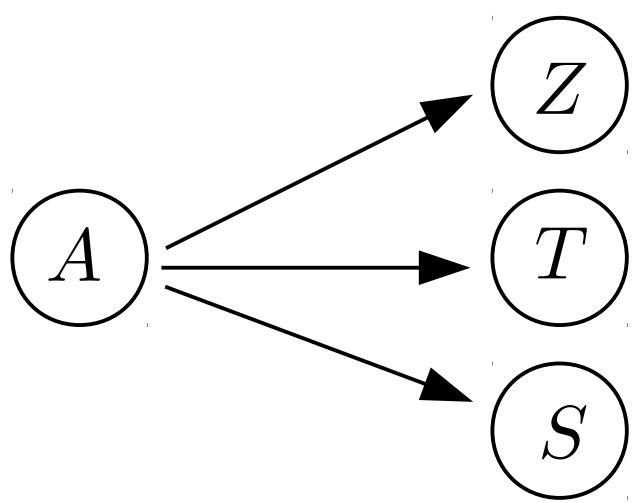




Successors of *A*

A search tree



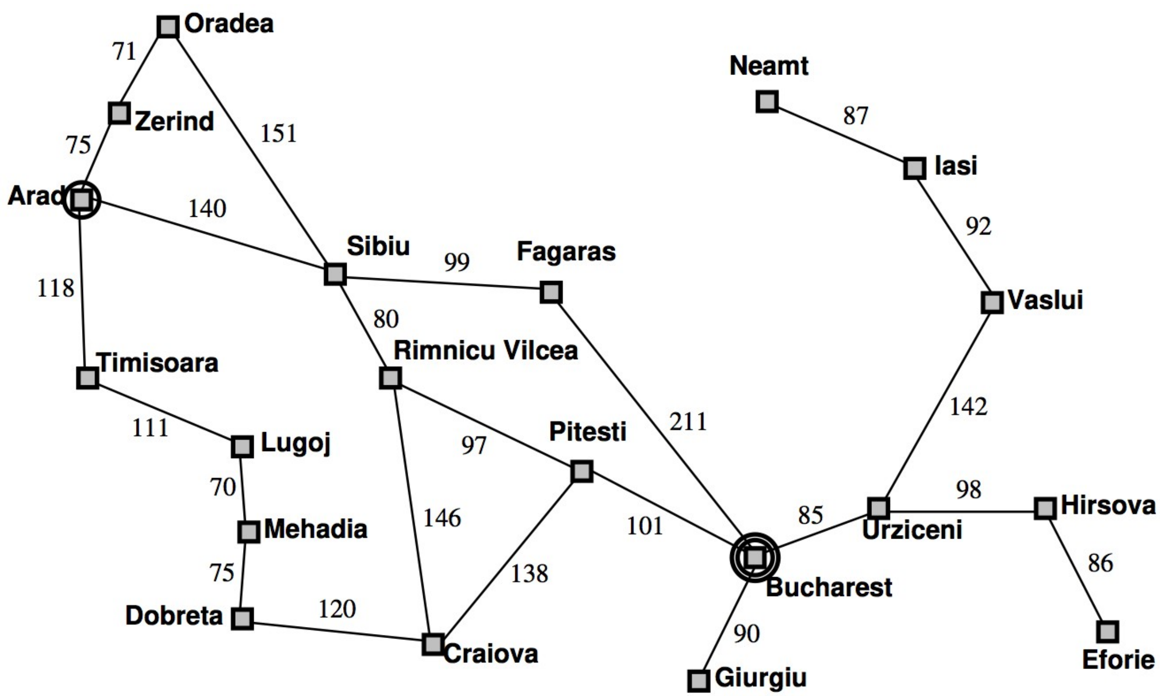


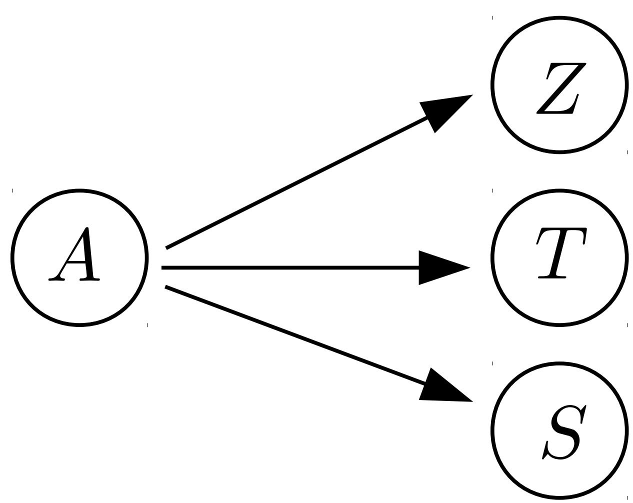
Successors of *A*

parent

children

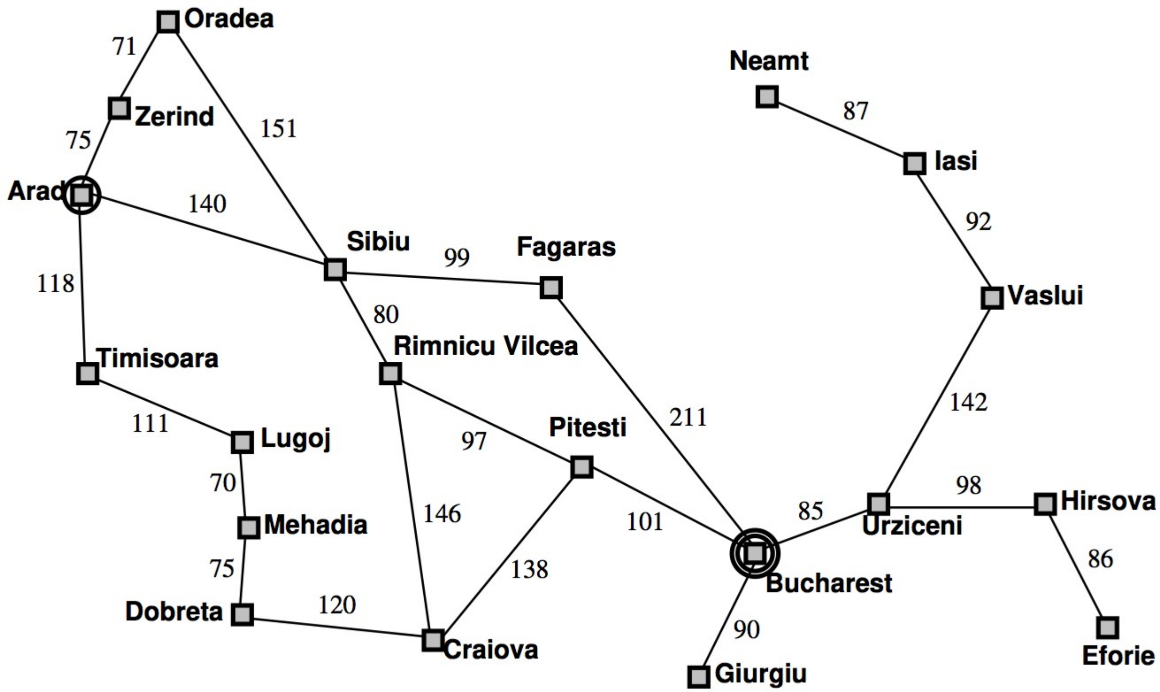
A search tree

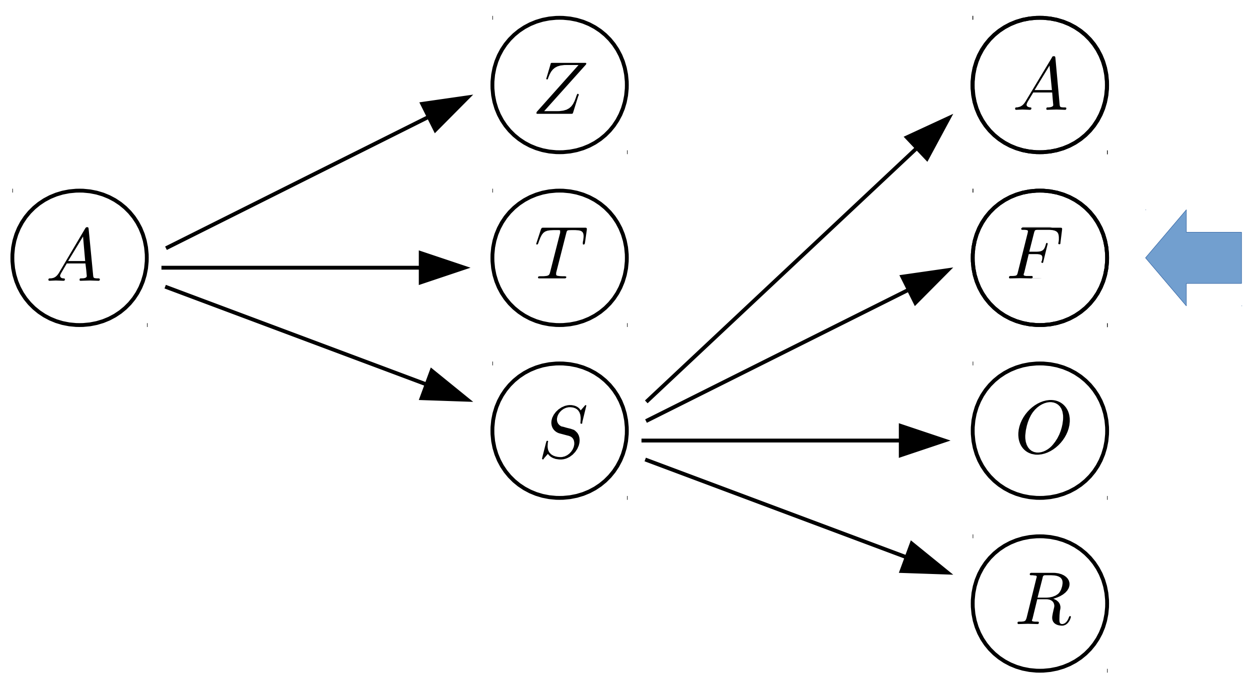




Let's expand *S* next

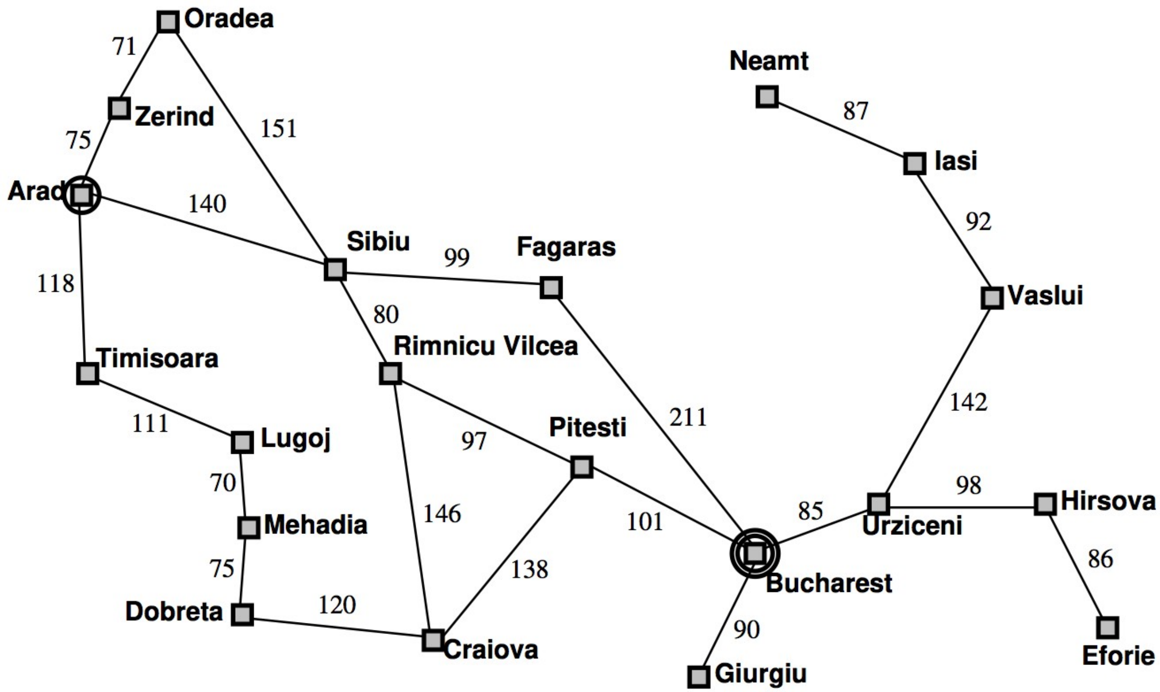
A search tree

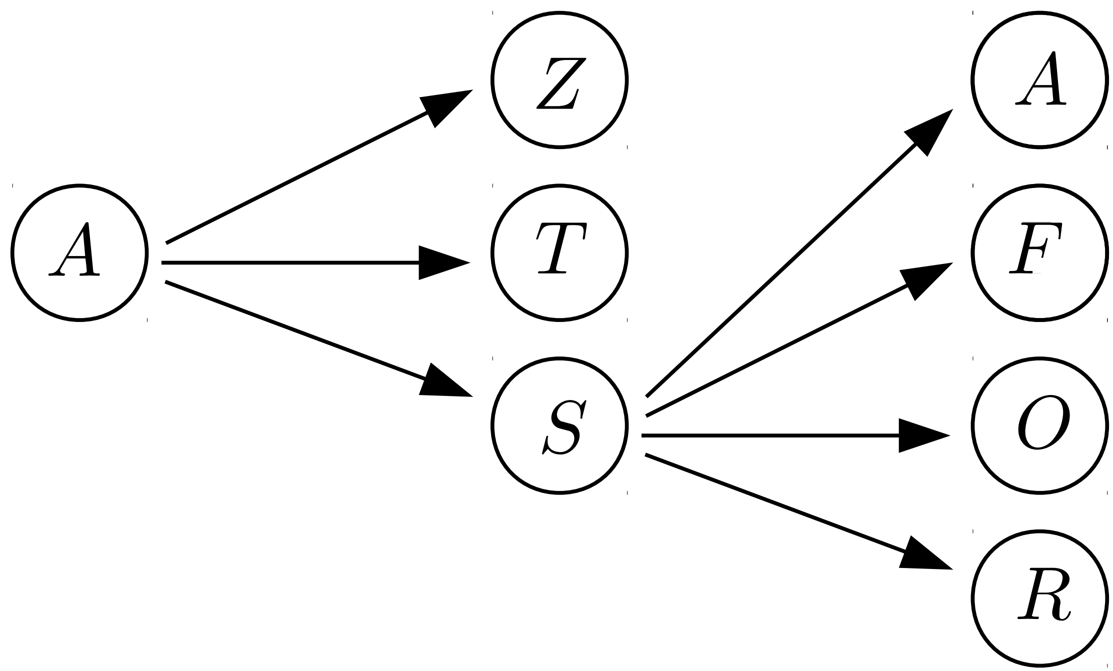




Successors of *S*

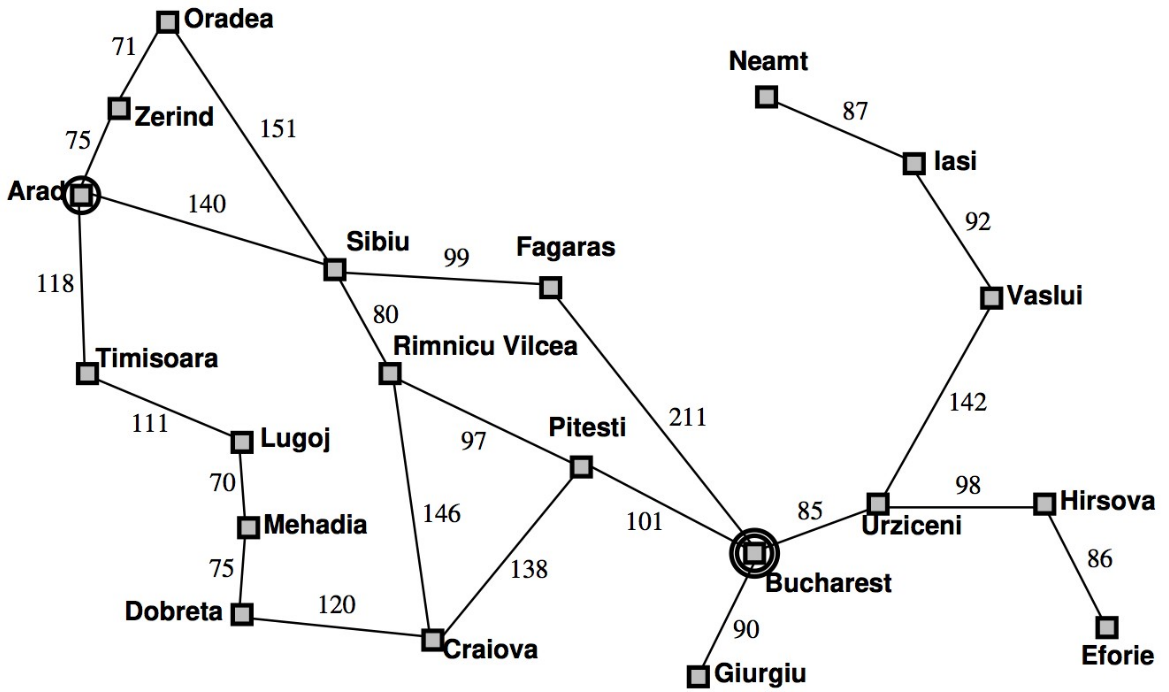
A search tree

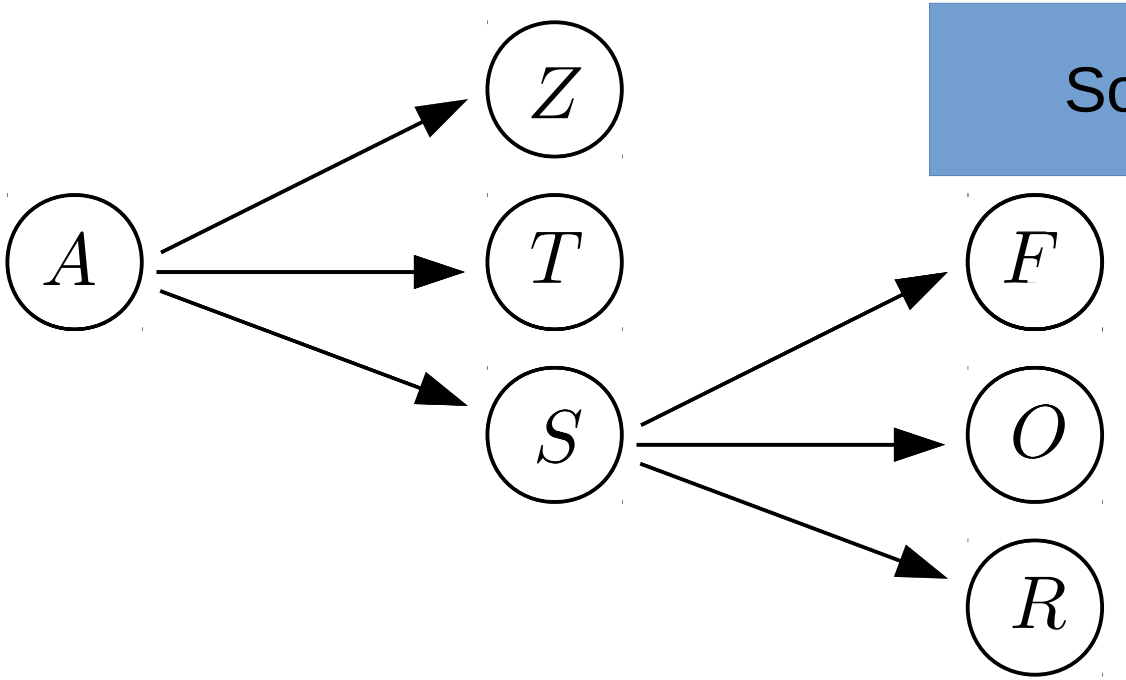




A was already visited!

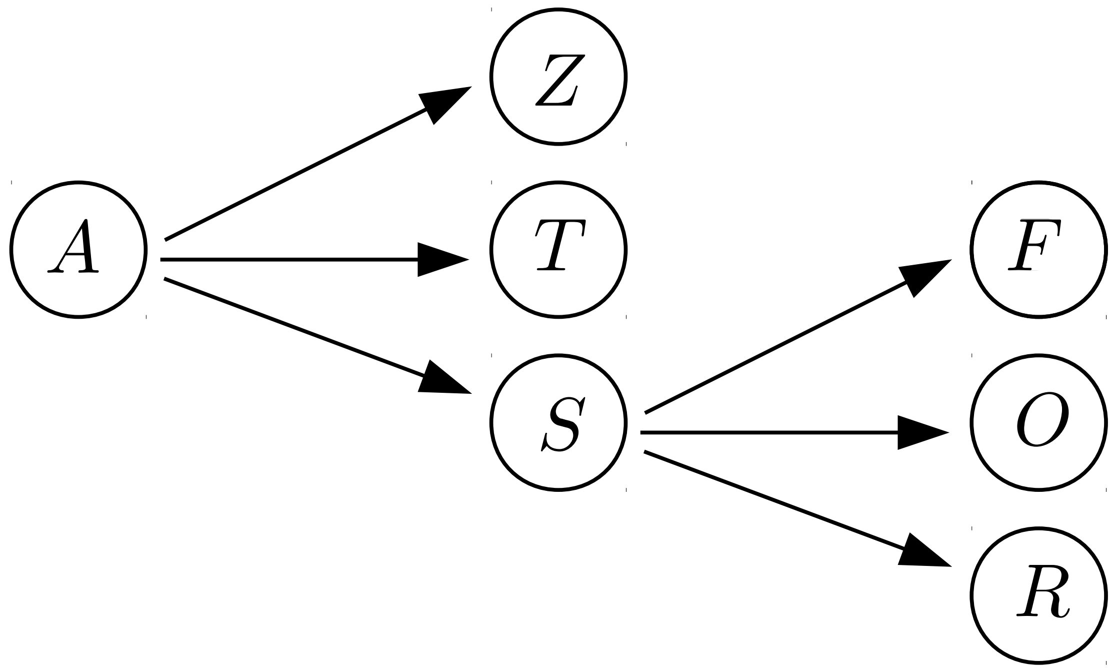
A search tree





So, prune it!

A search tree



<u>In what order should we expand states?</u>

– here, we expanded *S*, but we could also have expanded *Z* or *T*

– different search algorithms expand in different orders

Breadth first search (BFS)

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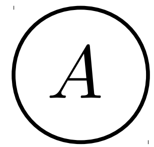

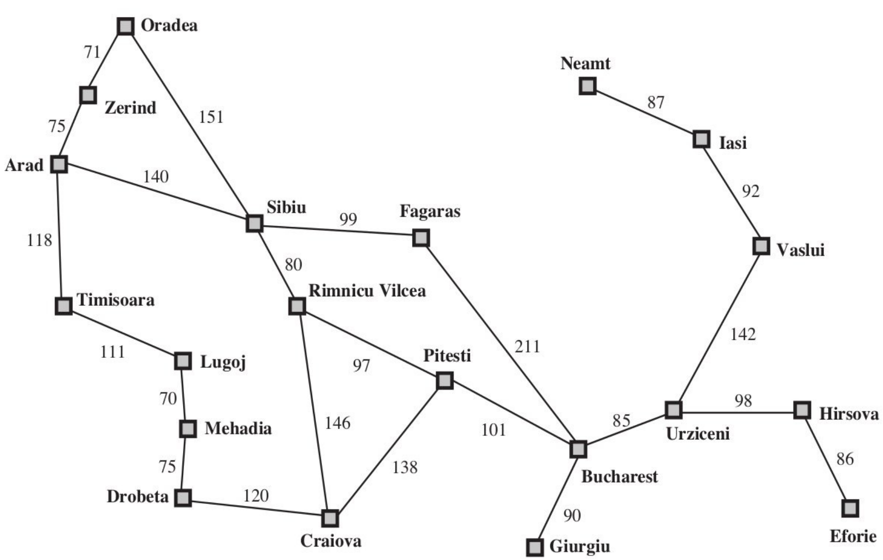

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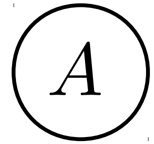
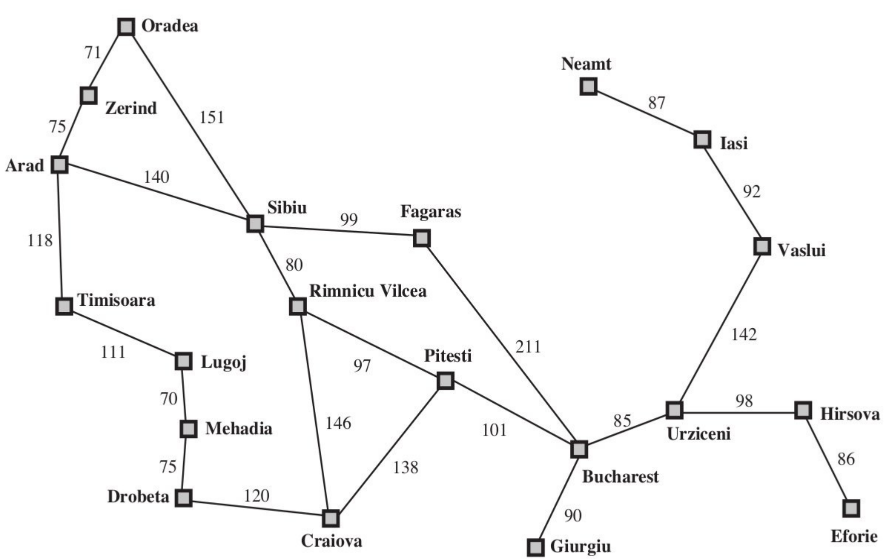


Start node



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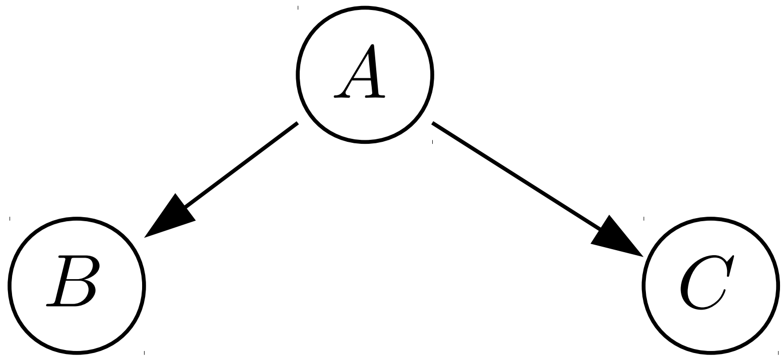

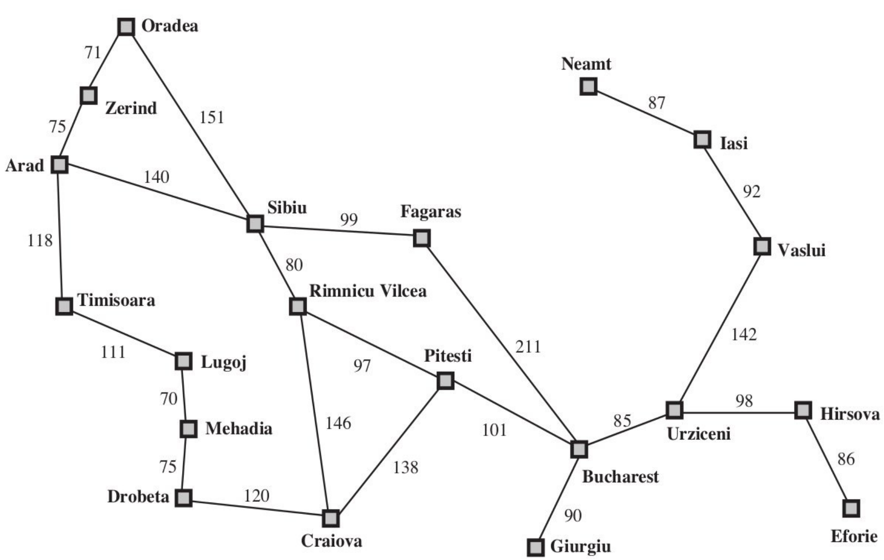

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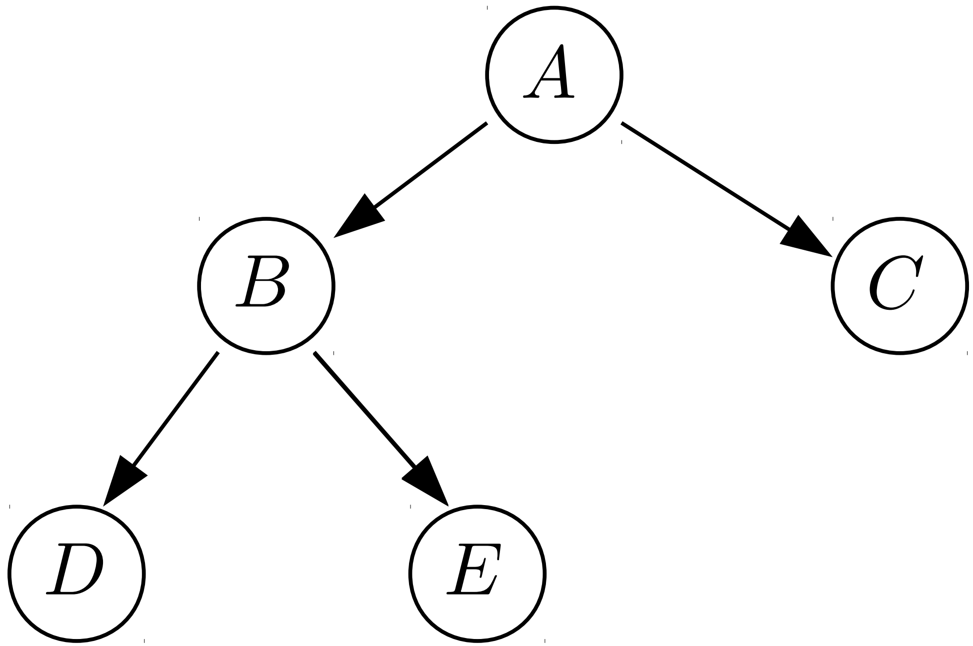

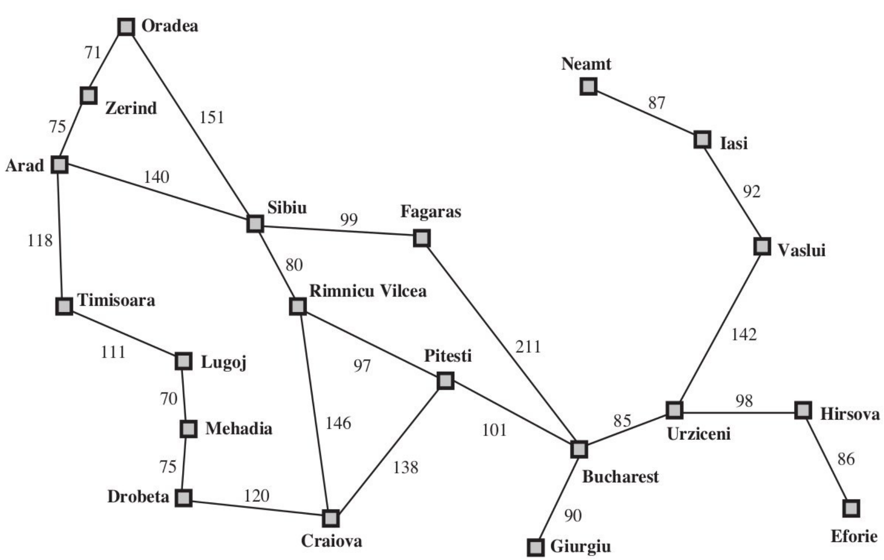

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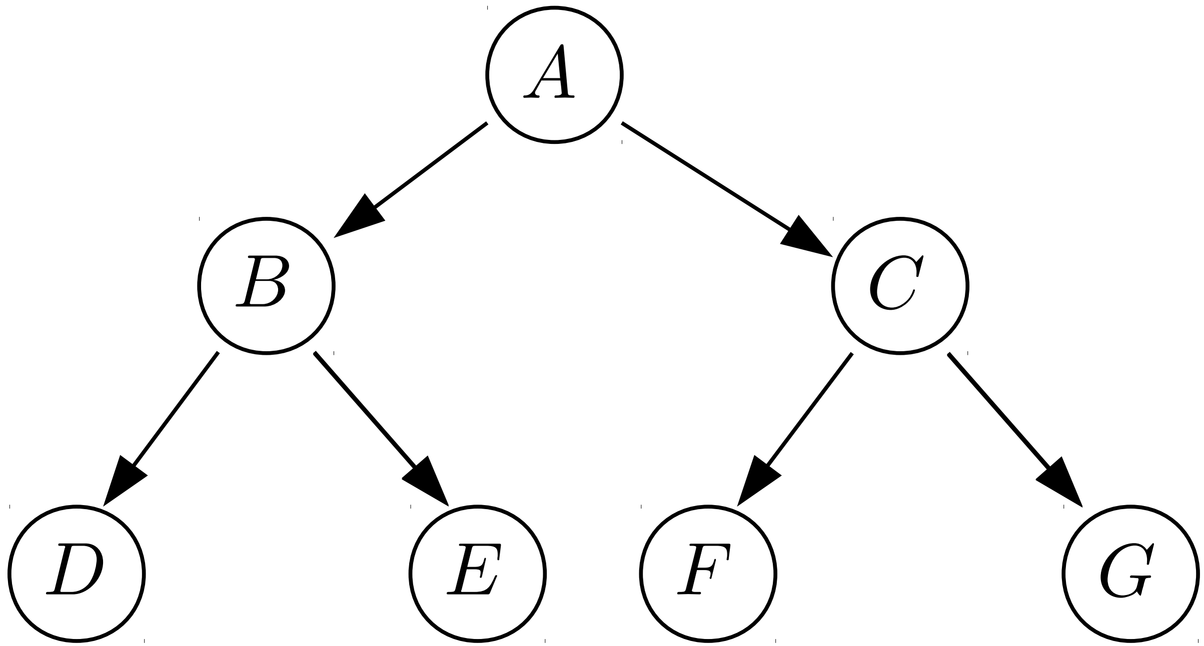
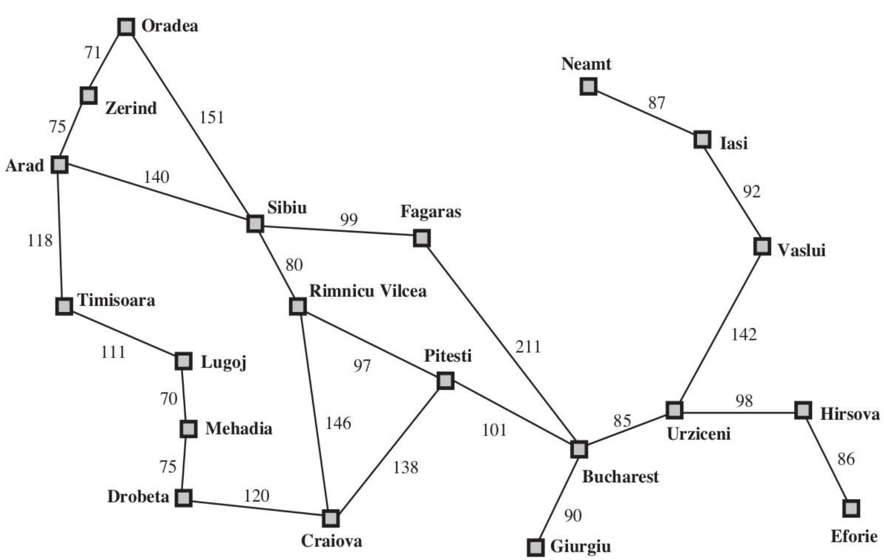




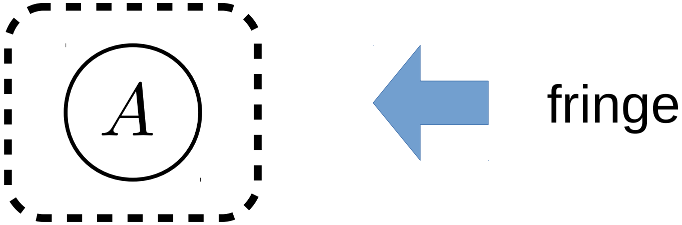
Breadth first search (BFS)

Fringe

We're going to maintain a queue called the fringe

– initialize the fringe as an empty queue

Breadth first search (BFS)

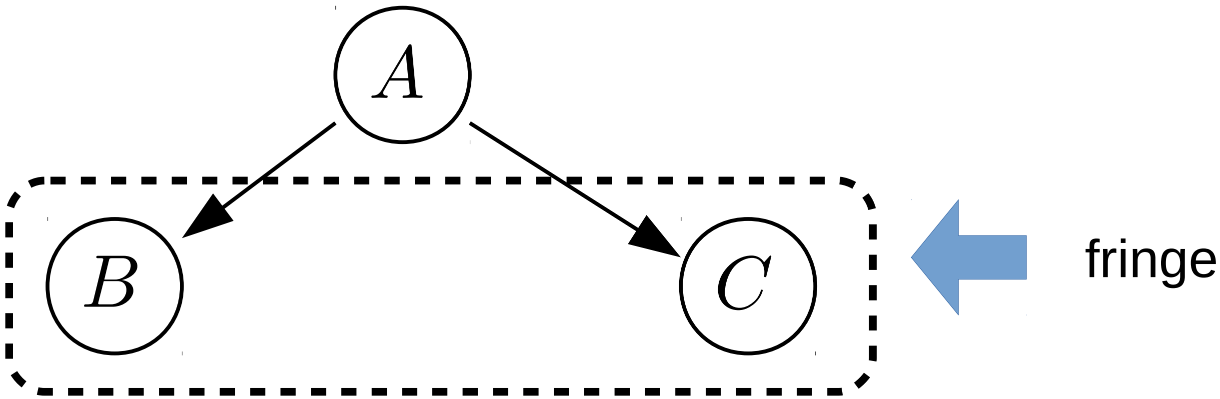


<u>Fringe</u>
A

– add *A* to the fringe

Breadth first search (BFS)

<u>Fringe</u>
B
C

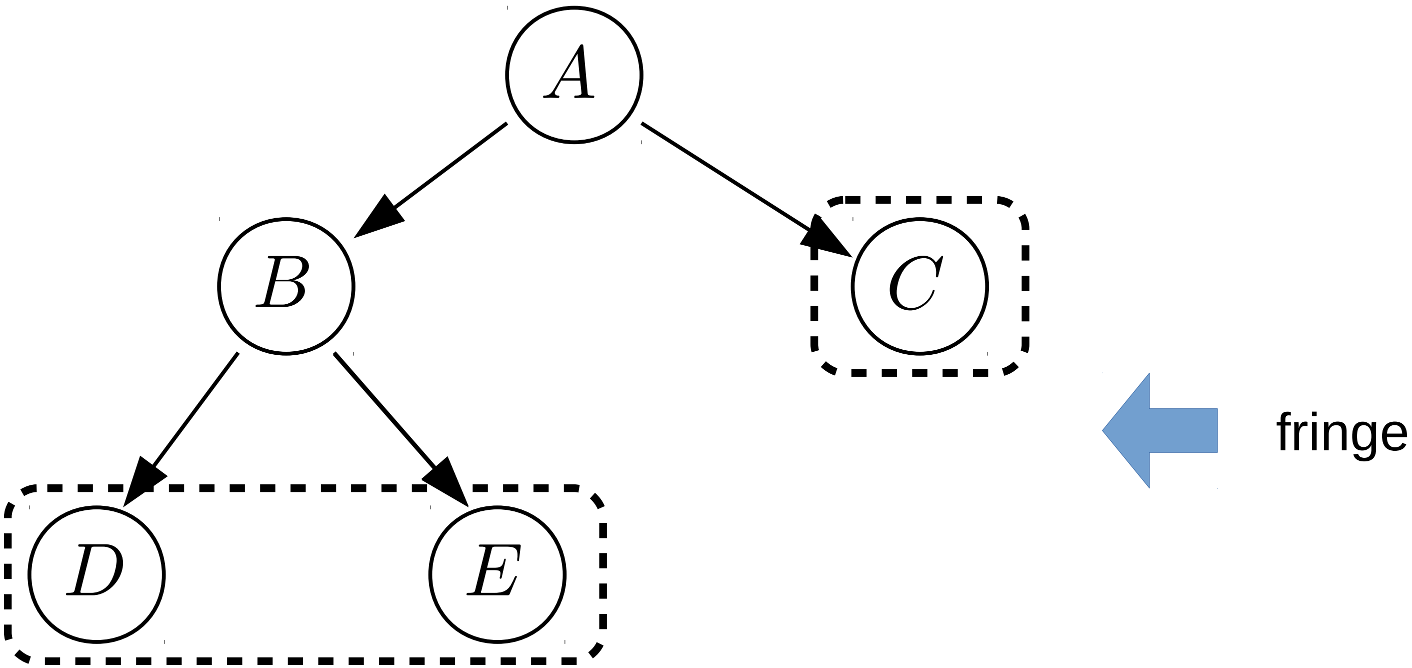


-- remove *A* from the fringe

-- add successors of *A* to the fringe

Breadth first search (BFS)

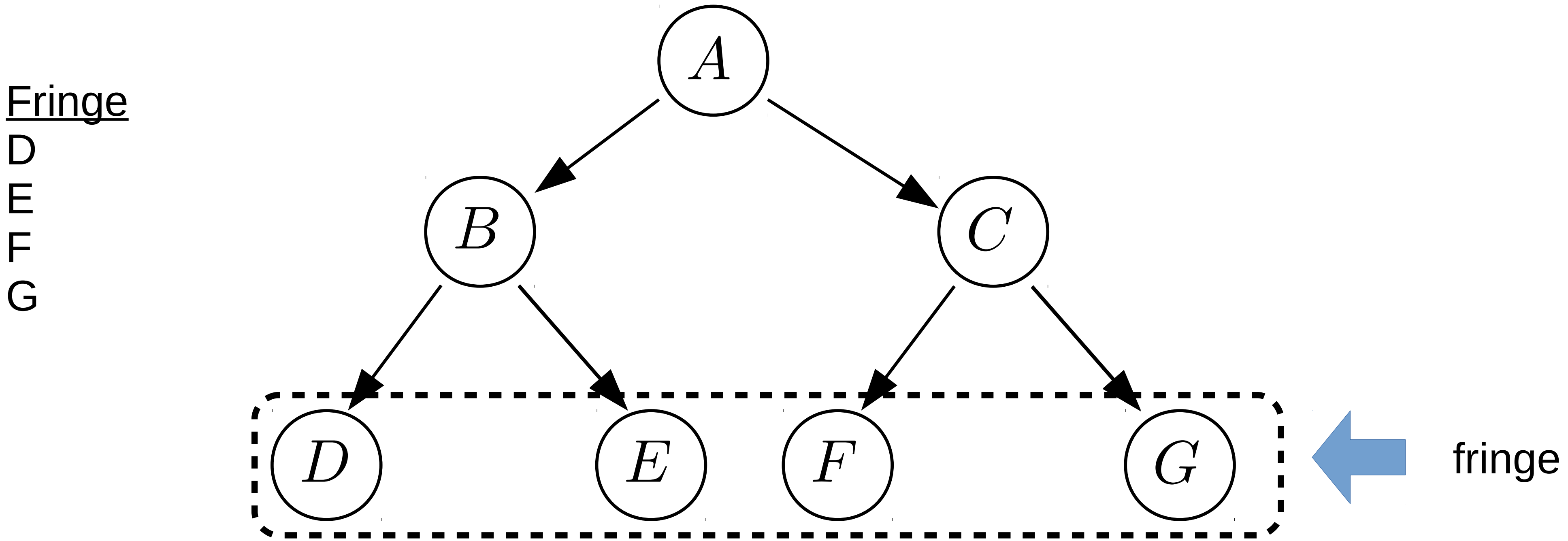
<u>Fringe</u>
C
D
E



-- remove *B* from the fringe

-- add successors of *B* to the fringe

Breadth first search (BFS)

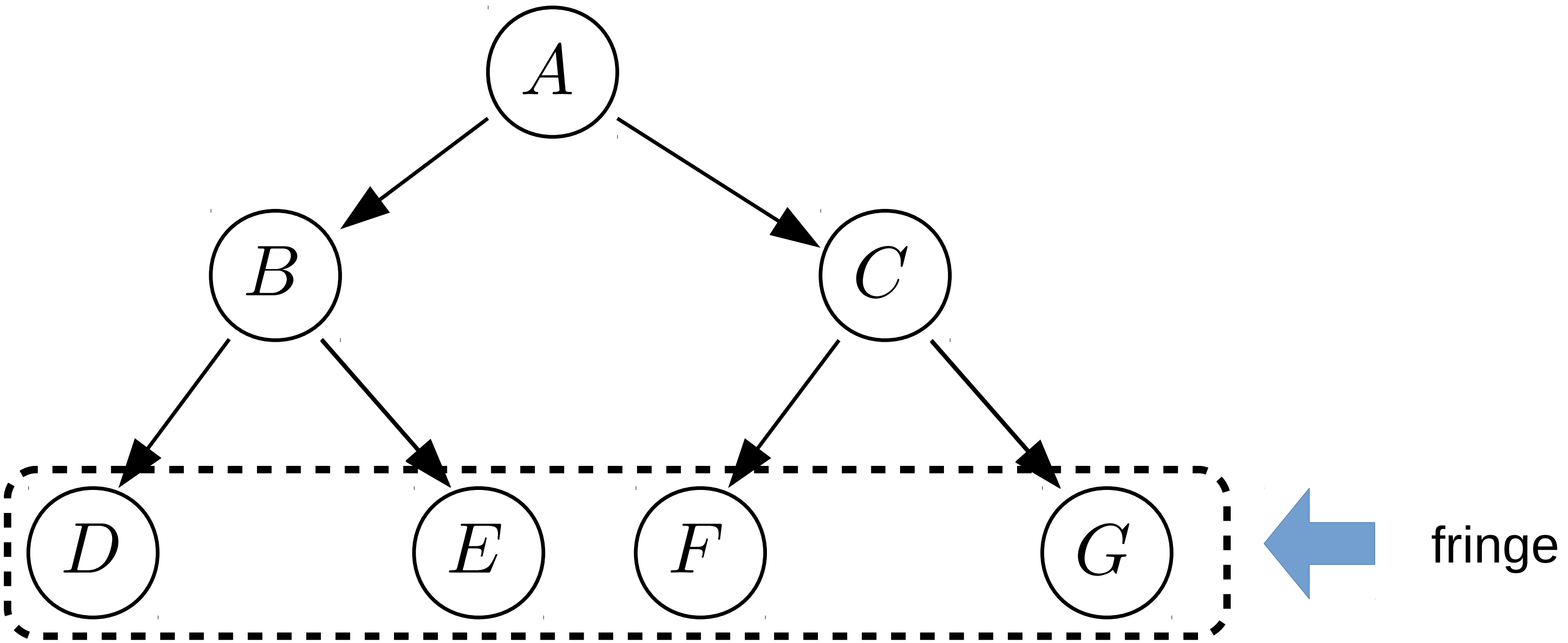


-- remove *C* from the fringe

-- add successors of *C* to the fringe

Breadth first search (BFS)

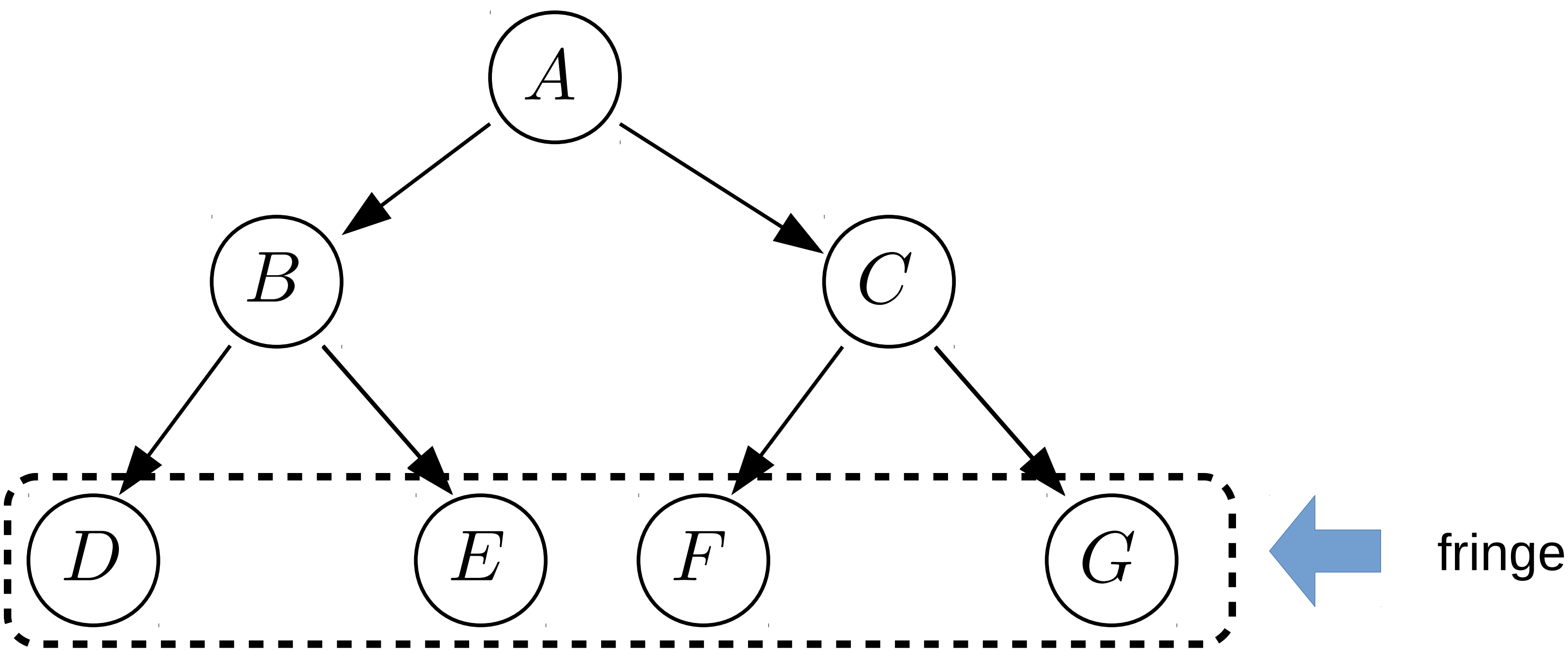
<u>Fringe</u>
D
E
F
G



Which state gets removed next from the fringe?

Breadth first search (BFS)

Fringe
D
E
F
G



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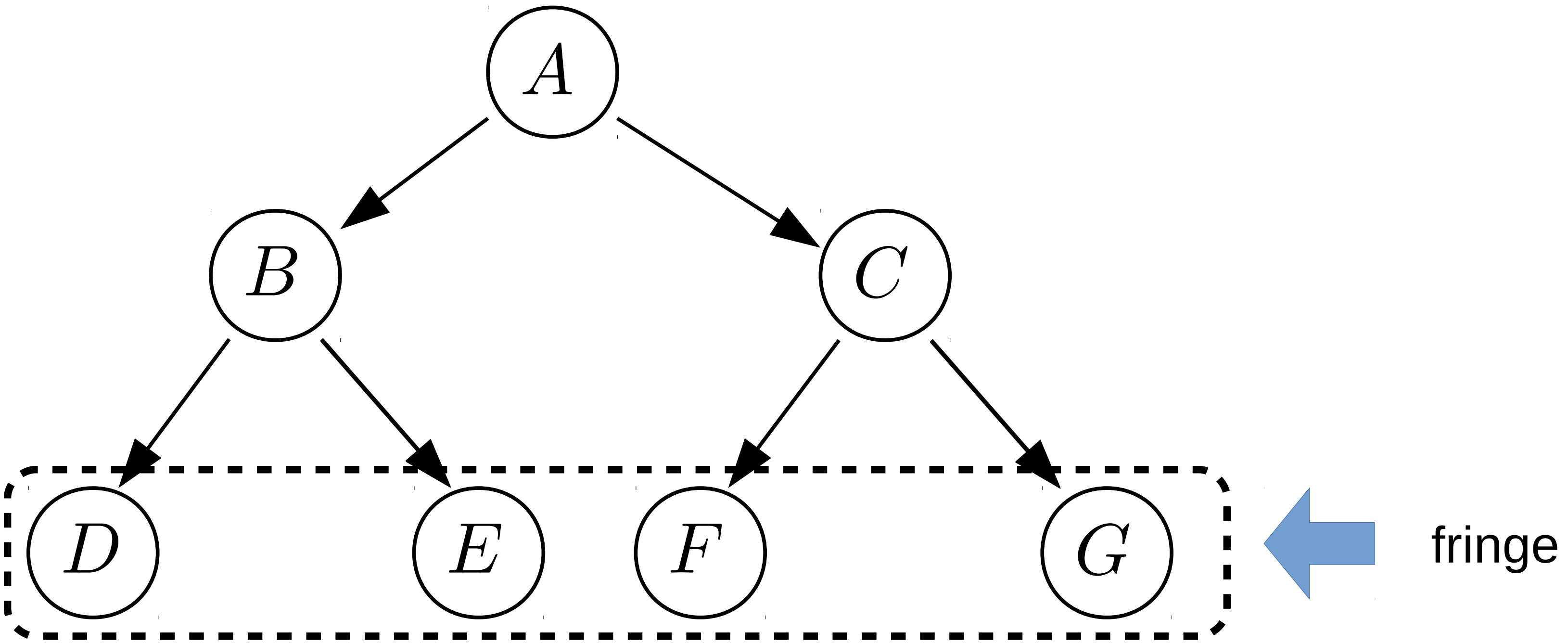
What kind of a queue is this?

Breadth first search (BFS)

<u>Fringe</u>
D
E
F
G



Which state gets removed next from the fringe?

What kind of a queue is this?

FIFO Queue!
(first in first out)

Breadth first search (BFS)

```
function BREADTH-FIRST-SEARCH(problem) returns a solution, or failure
  node ← a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  frontier ← a FIFO queue with node as the only element
  explored ← an empty set
  loop do
      if EMPTY?(frontier) then return failure
      node ← POP(frontier)   /* chooses the shallowest node in frontier */
      add node.STATE to explored
      for each action in problem.ACTIONS(node.STATE) do
          child ← CHILD-NODE(problem, node, action)
          if child.STATE is not in explored or frontier then
              if problem.GOAL-TEST(child.STATE) then return SOLUTION(child)
              frontier ← INSERT(child, frontier)
```

Figure 3.11 Breadth-first search on a graph.

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Figure 3.11 Breadth-first search on a graph.

What is the purpose of the *explored* set?

BFS Properties

Is BFS <u>complete</u>?
– is it guaranteed to find a solution if one exists?

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– how many states are expanded before finding a sol'n?
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  – d: depth of shallowest solution
  – complexity = ???

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– how much memory is required?
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Is BFS optimal?
– is it guaranteed to find the best solution (shortest path)?

Uniform Cost Search (UCS)

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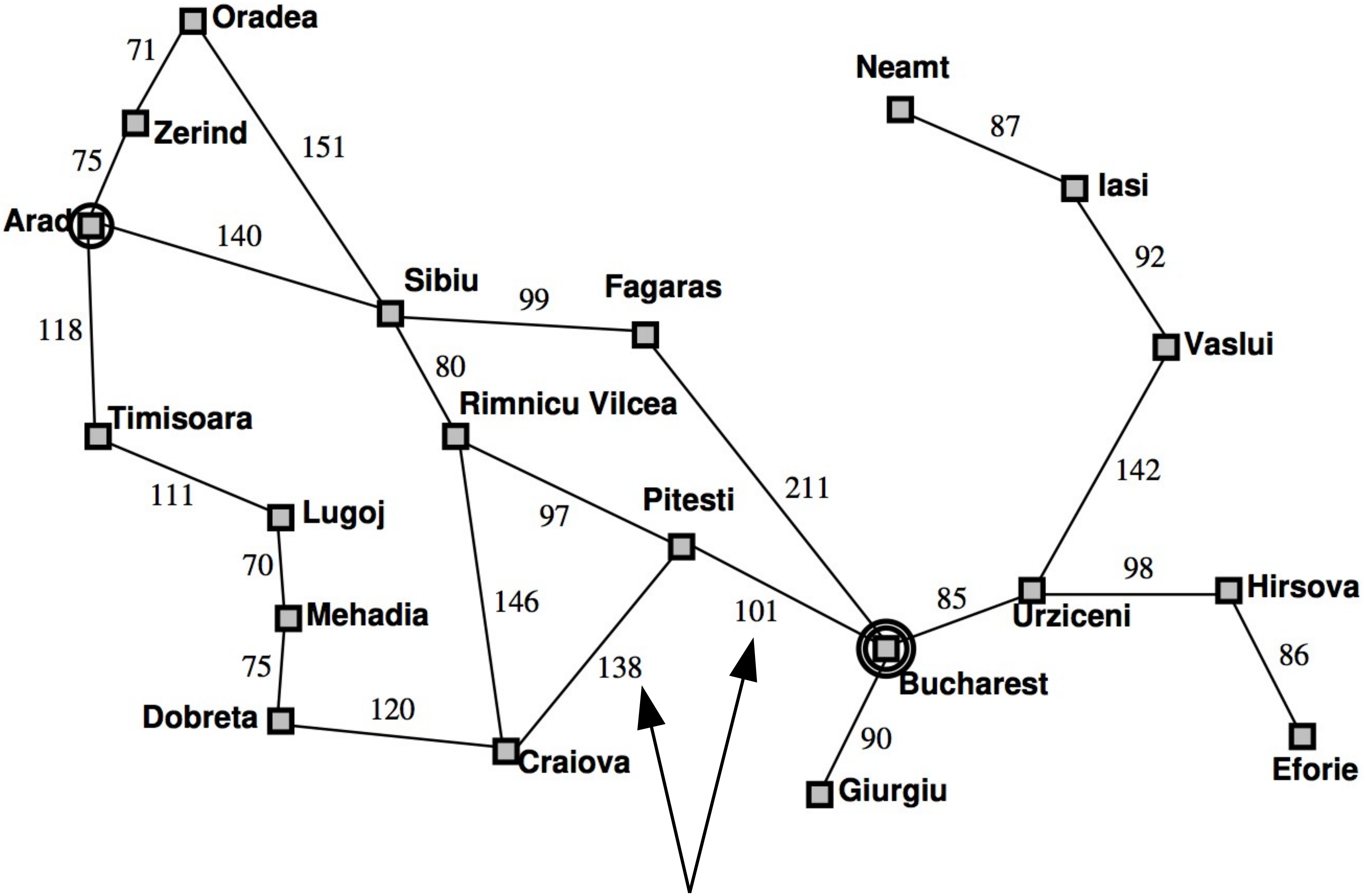
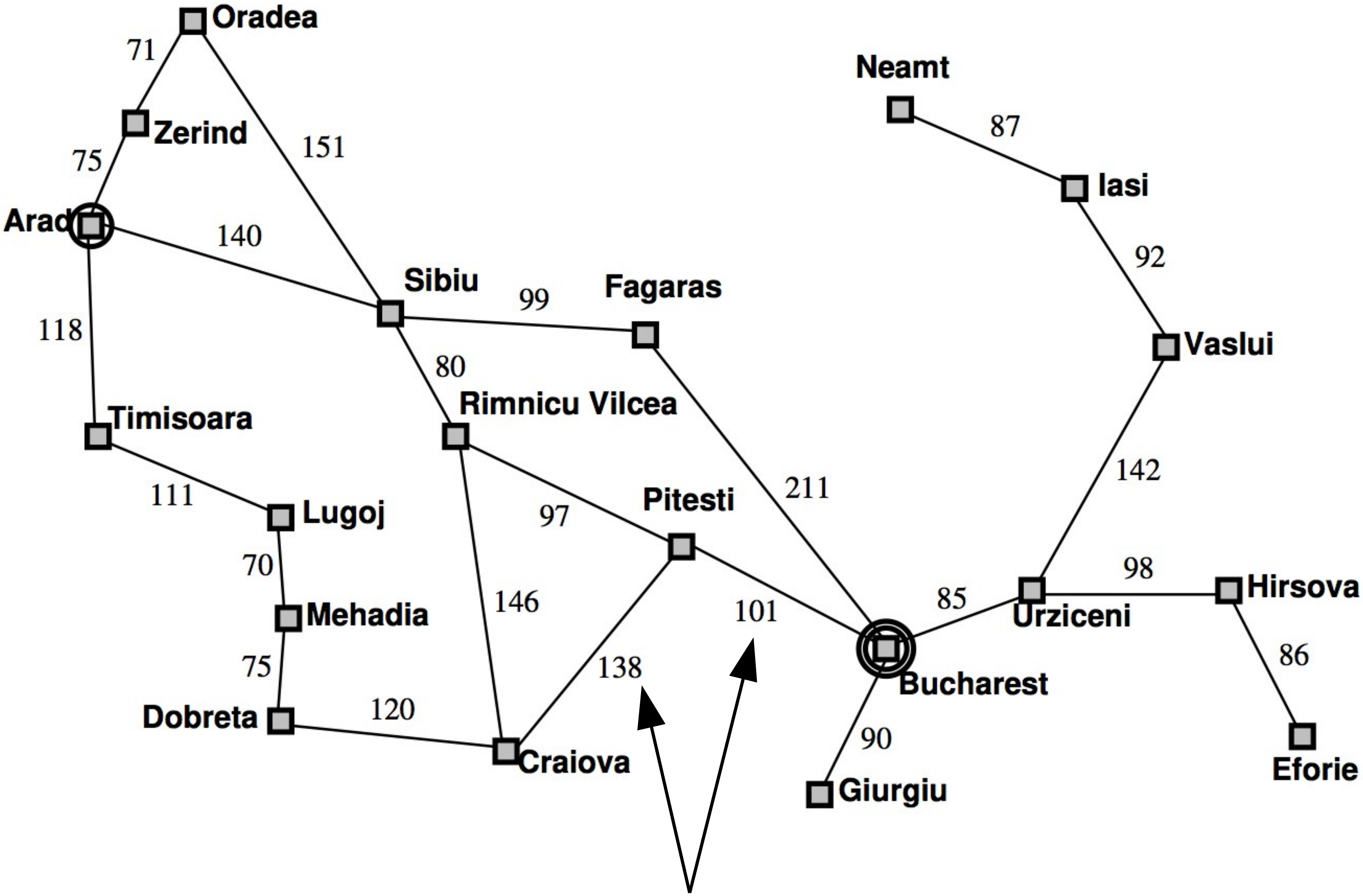


Notice the distances between cities

Uniform Cost Search (UCS)



Notice the distances between cities
– does BFS take these distances into account?

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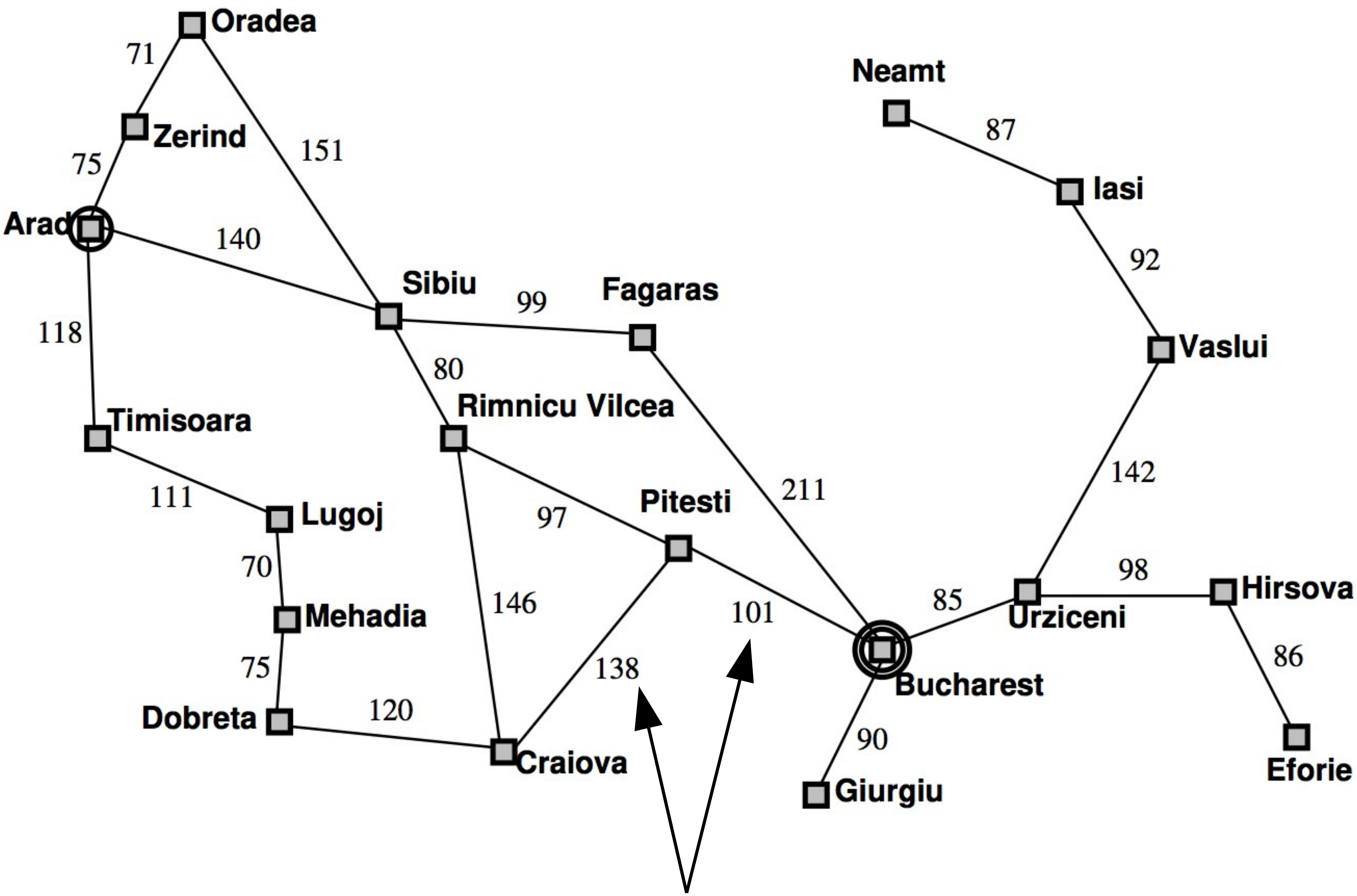
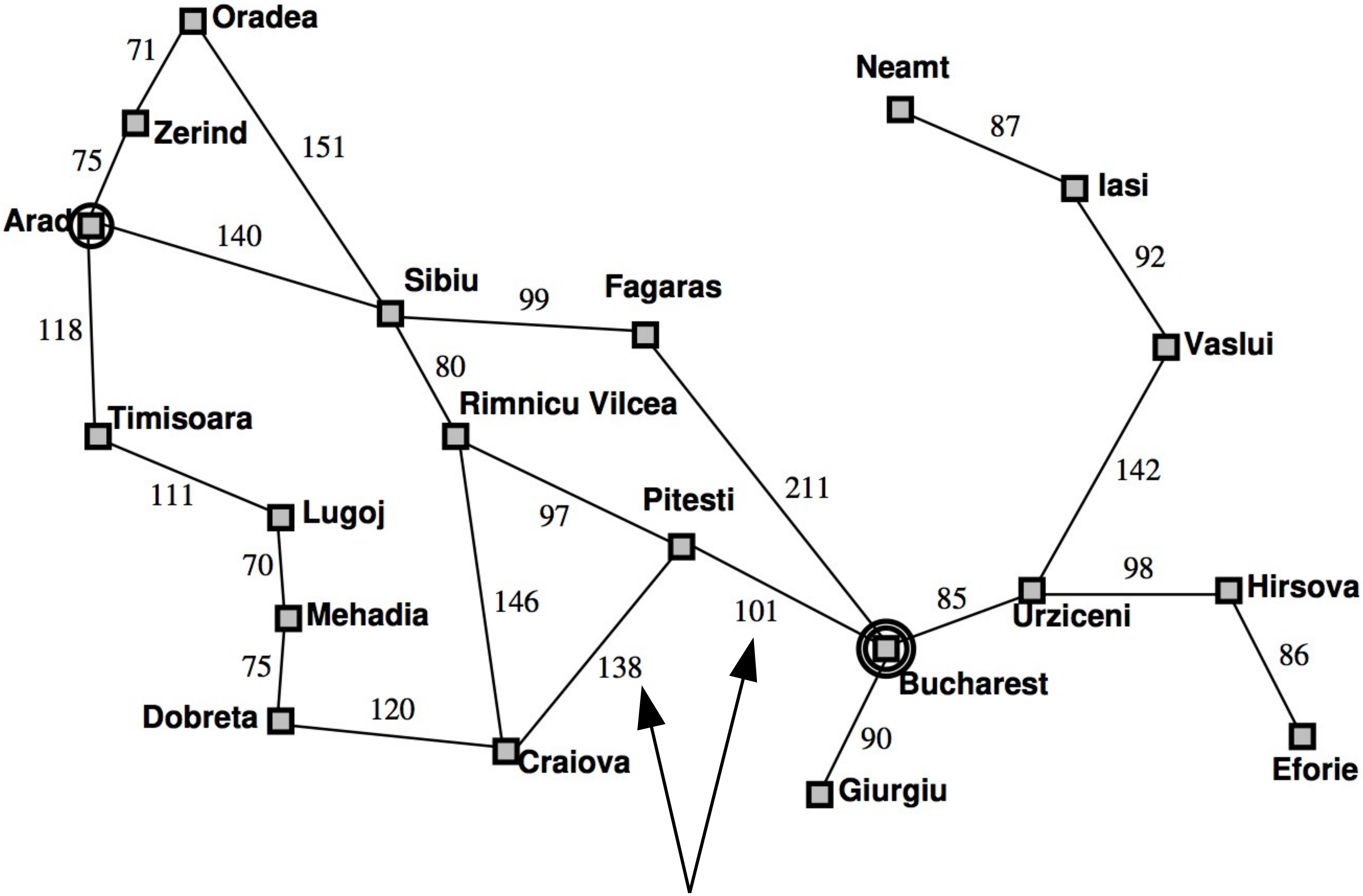


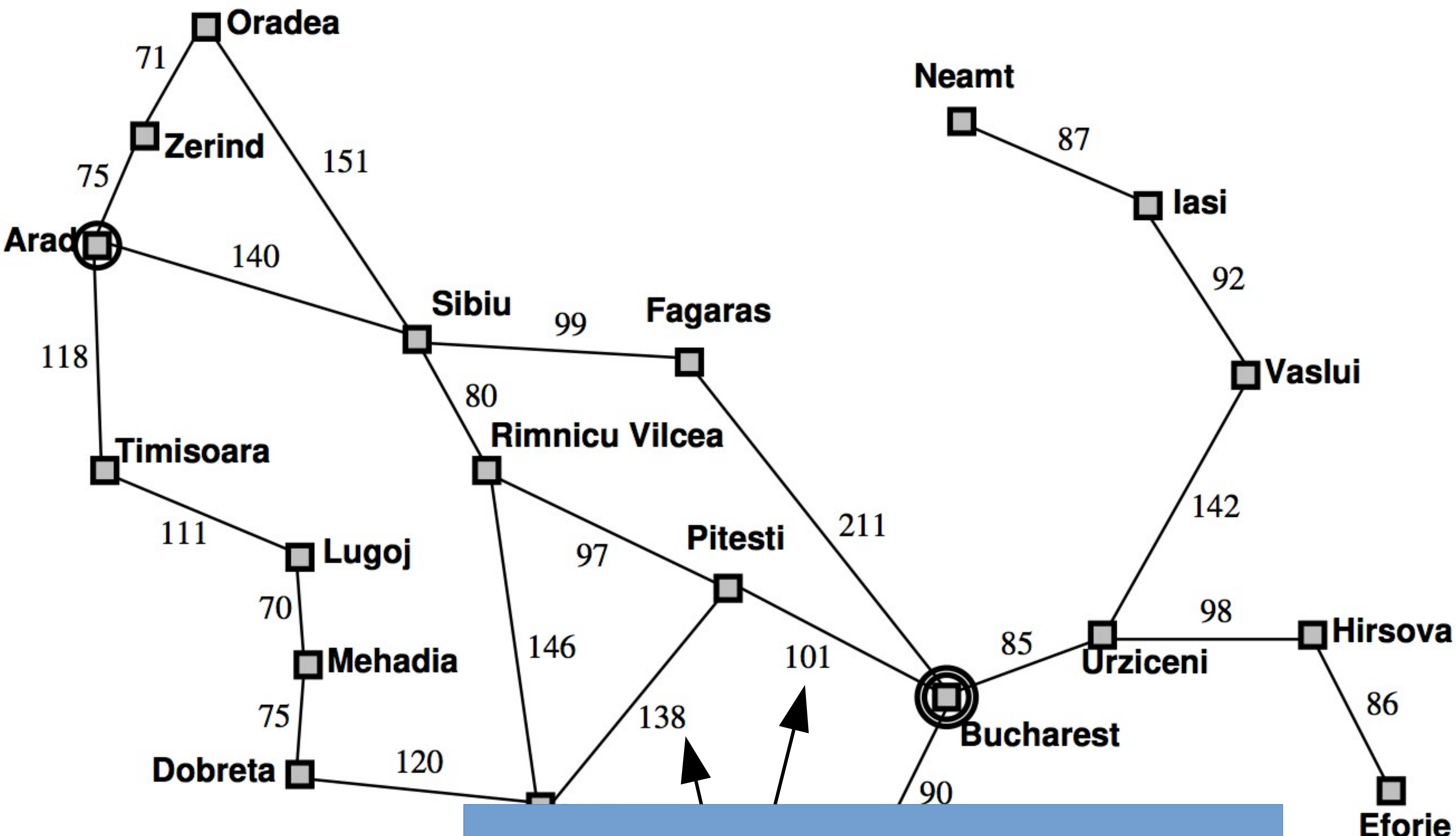
Notice the distances between cities
– does BFS take these distances into account?
– does BFS find the path w/ shortest milage?

Uniform Cost Search (UCS)



Notice the distances between cities
– does BFS take these distances into account?
– does BFS find the path w/ shortest milage?
– compare S-F-B with S-R-P-B. Which costs less?

Uniform Cost Search (UCS)



How do we fix this?

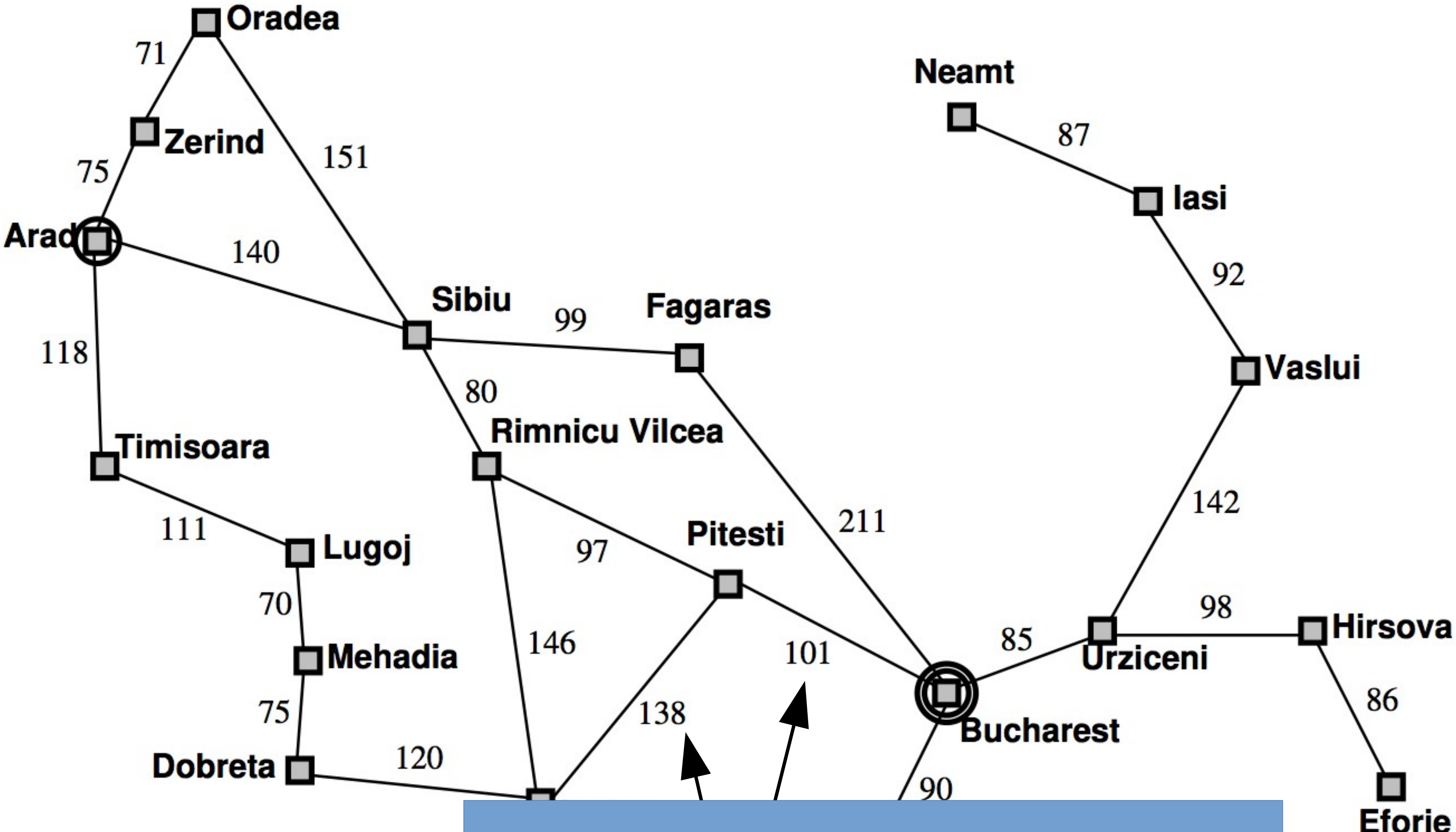
Notic

– do unt?

– do ?

– co s less?

Uniform Cost Search (UCS)



How do we fix this?
<u>UCS!</u>

Notic

– do … unt?

– do … ?

– co … s less?

Uniform Cost Search (UCS)

Same as BFS except: expand node w/ smallest <u>path cost</u>

Length of path

Uniform Cost Search (UCS)

Same as BFS except: expand node w/ smallest <u>path cost</u>

Length of path

Cost of going from state *A* to *B*: $c(A, B)$

Minimum cost of path going from start state to *B*: $g(B)$

Uniform Cost Search (UCS)

Same as BFS except: expand node w/ smallest path cost

Length of path

Cost of going from state *A* to *B:* $c(A, B)$

Minimum cost of path going from start state to *B:* $g(B)$

BFS: expands states in order of hops from start

UCS: expands states in order of $g(s)$

Uniform Cost Search (UCS)

Same as BFS except: expand node w/ smallest <u>path cost</u>

Length of path

Cost of going from state *A* to *B:* $c(A, B)$

Minimum cost of path going from start state to *B:* $g(B)$

BFS: exp

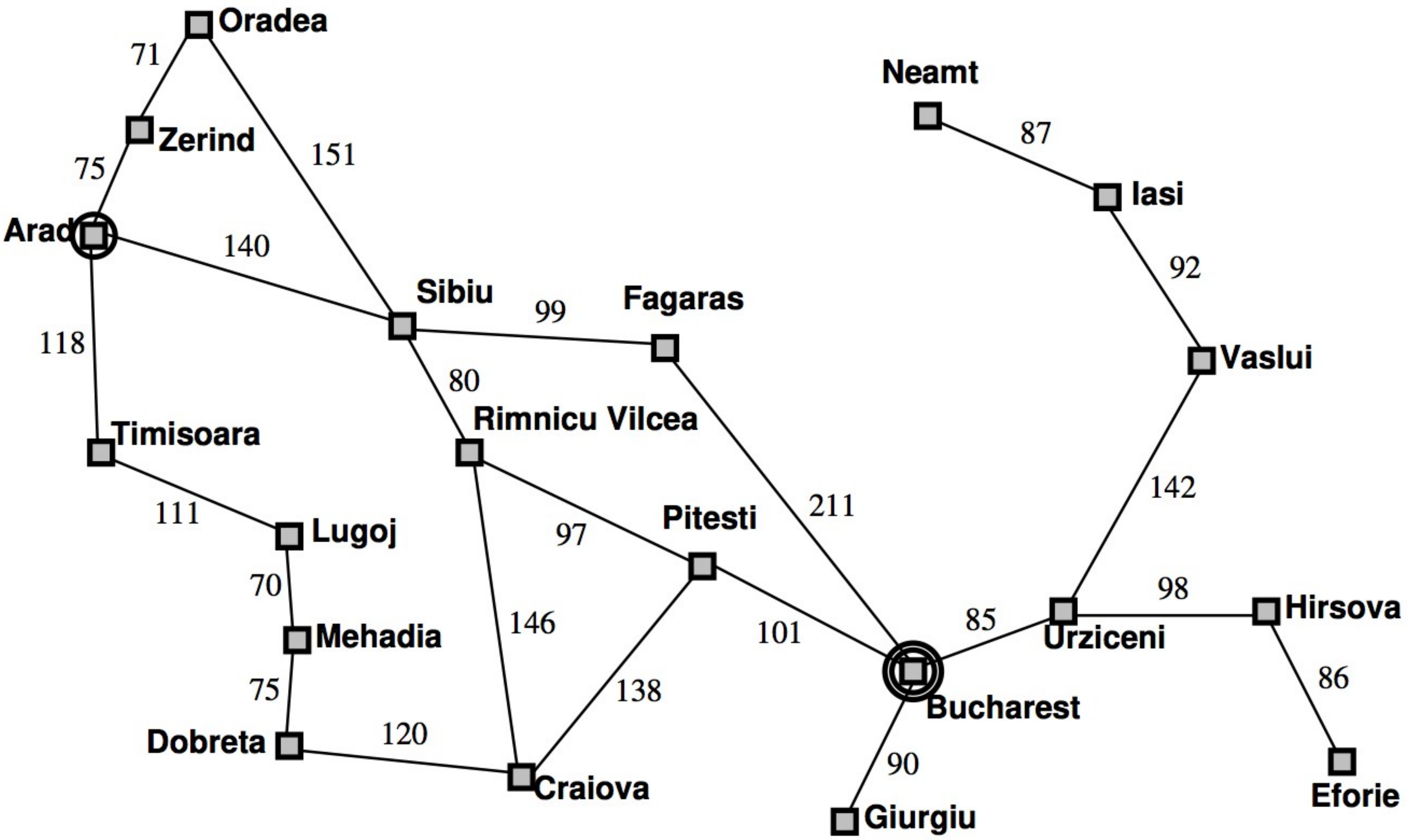
UCS: ex

How?

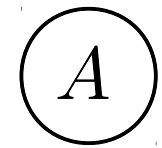
Uniform Cost Search (UCS)

Simple answer: change the FIFO to a priority queue
– the priority of each element in the queue is its path cost.

Uniform Cost Search (UCS)



UCS

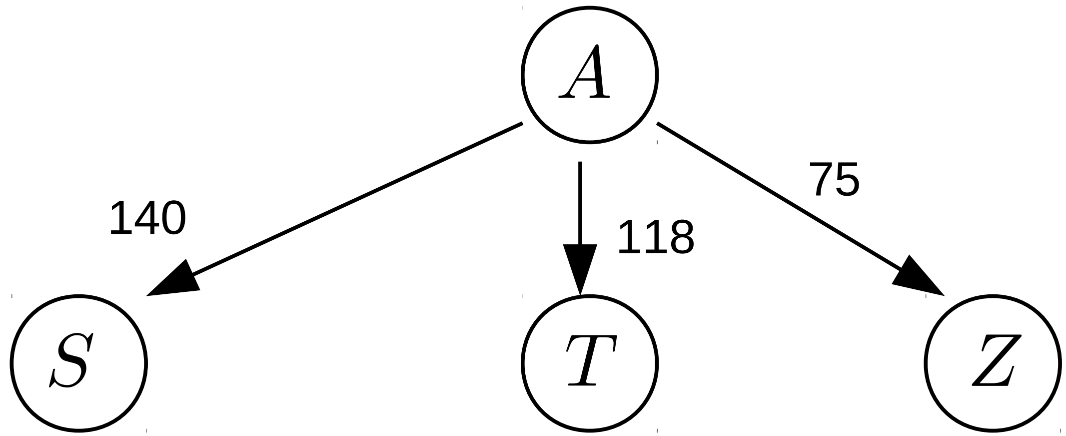


Fringe	Path Cost
A	0

Explored set:

UCS

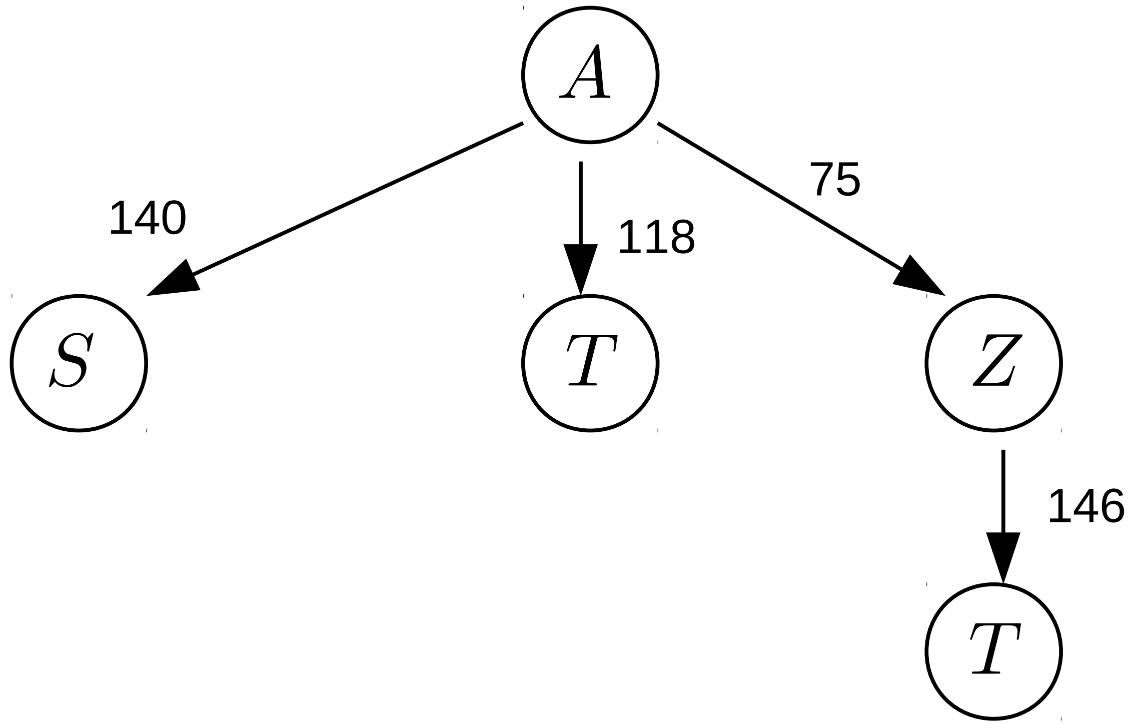
Fringe	Path Cost
~~A~~	~~0~~
S	140
T	118
Z	75



Explored set: A

UCS

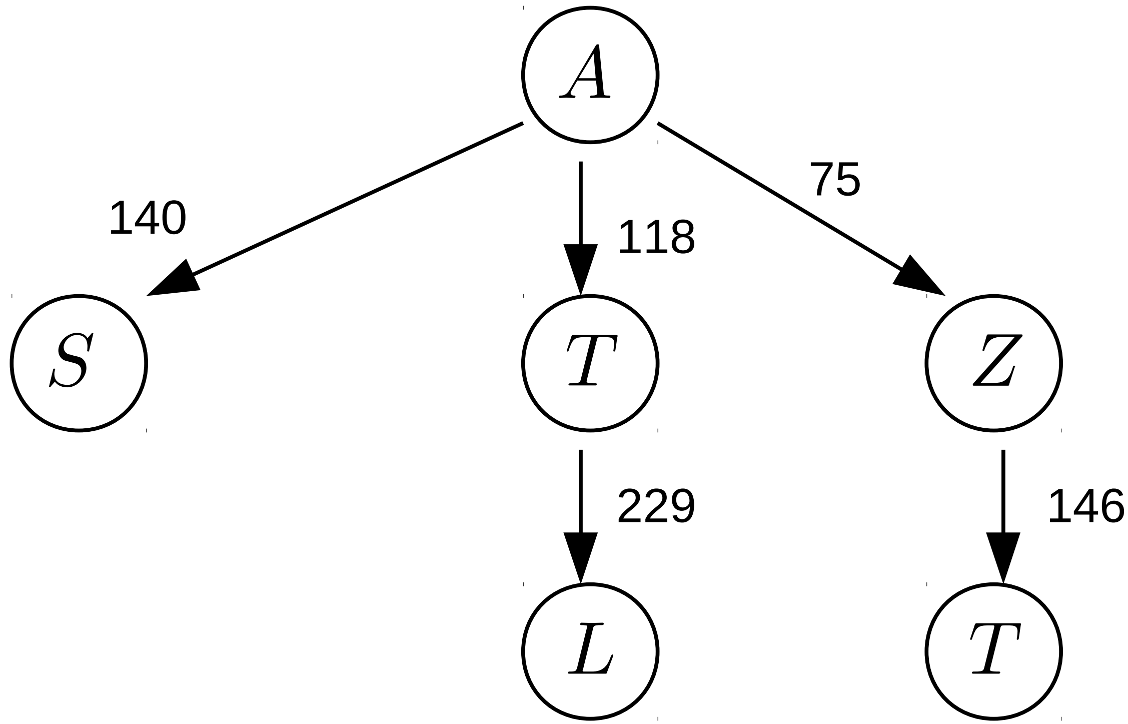
Fringe	Path Cost
~~A~~	~~0~~
S	140
T	118
~~Z~~	~~75~~
T	146



Explored set: A, Z

UCS

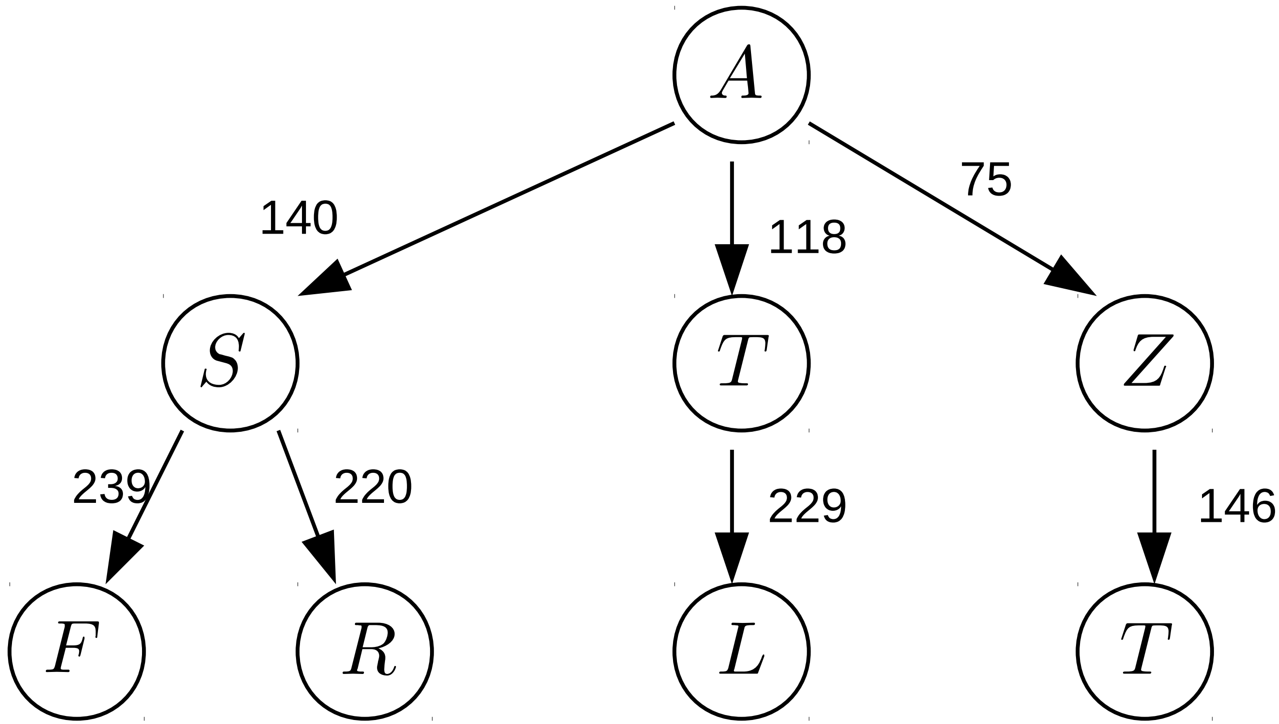
Fringe	Path Cost
~~A~~	~~0~~
S	140
~~T~~	~~118~~
~~Z~~	~~75~~
T	146
L	229



Explored set: A, Z, T

UCS

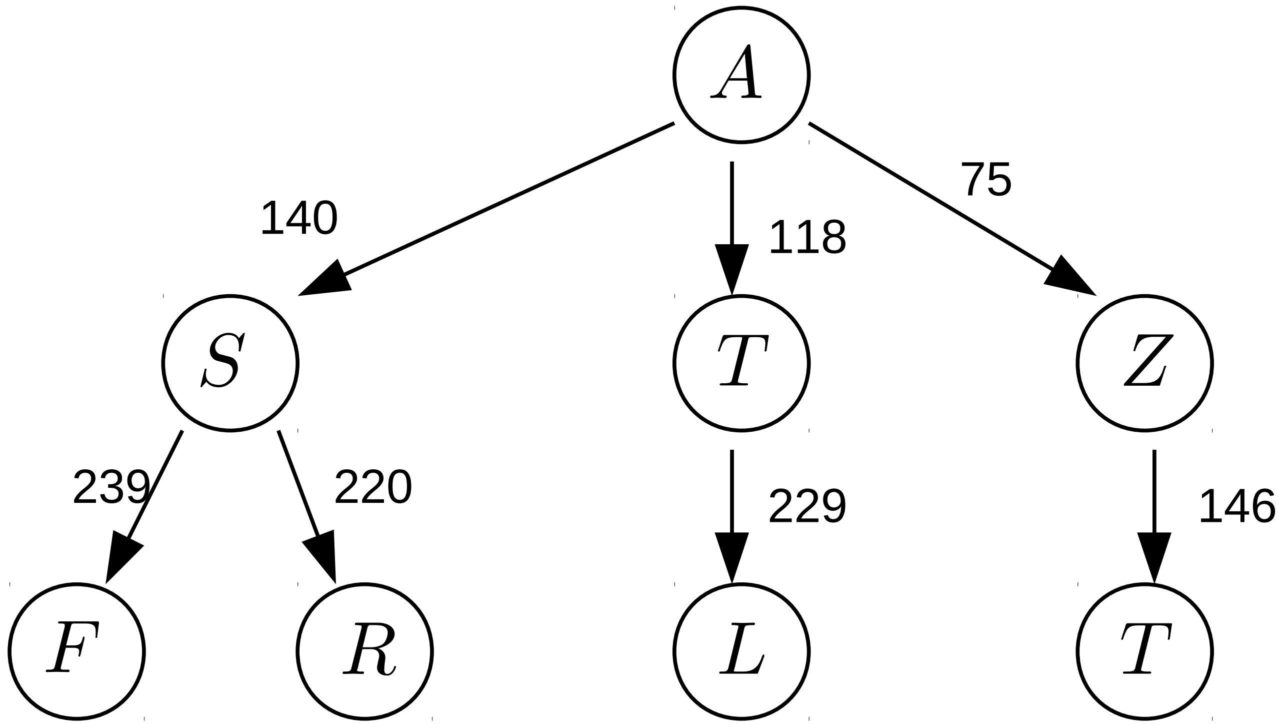
Fringe	Path Cost
~~A~~	~~0~~
~~S~~	~~140~~
~~T~~	~~118~~
~~Z~~	~~75~~
T	146
L	229
F	239
R	220



Explored set: A, Z, T, S

UCS

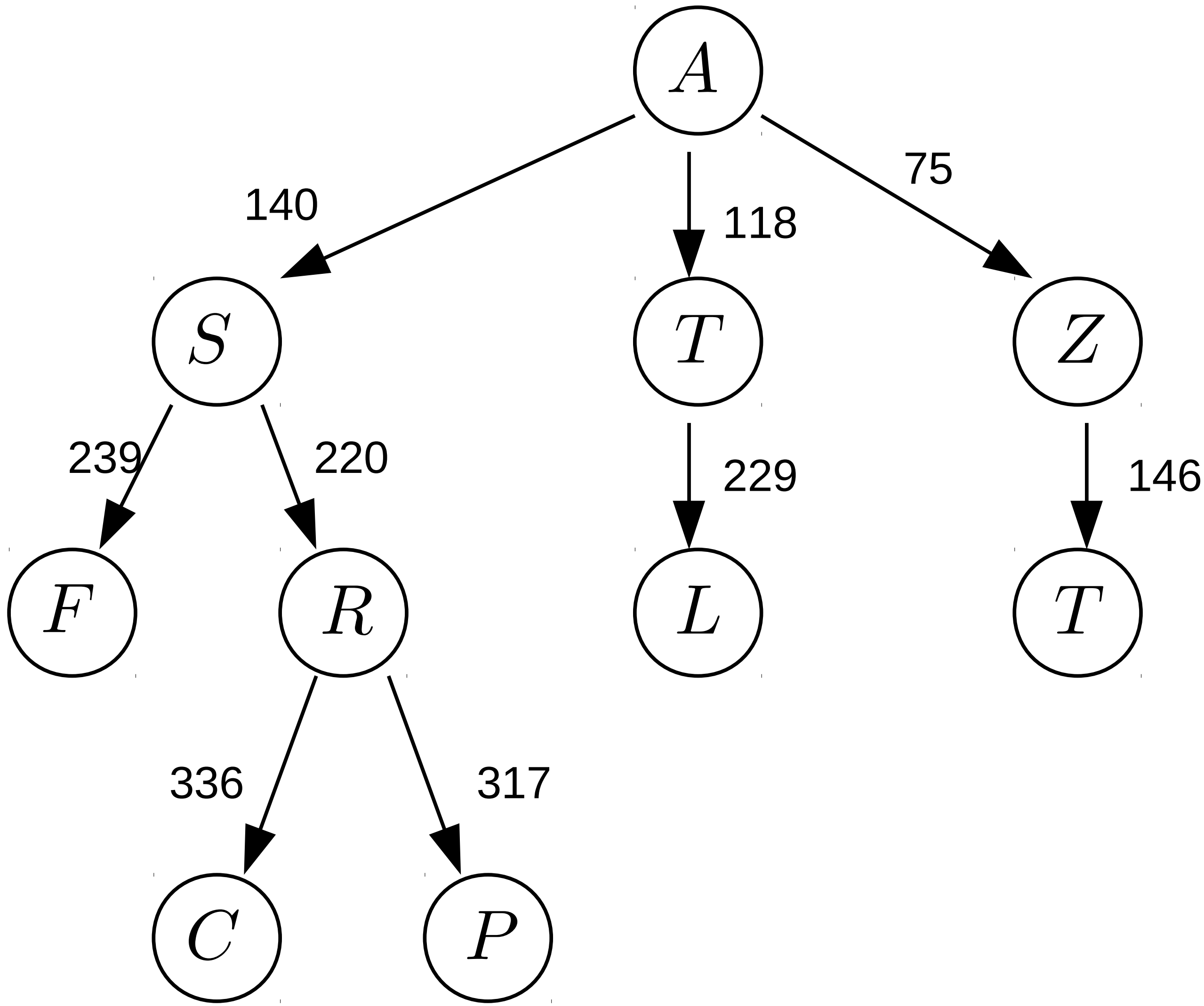
Fringe	Path Cost
~~A~~	~~0~~
~~S~~	~~140~~
~~T~~	~~118~~
~~Z~~	~~75~~
~~T~~	~~146~~
L	229
F	239
R	220



Explored set: A, Z, T, S

UCS

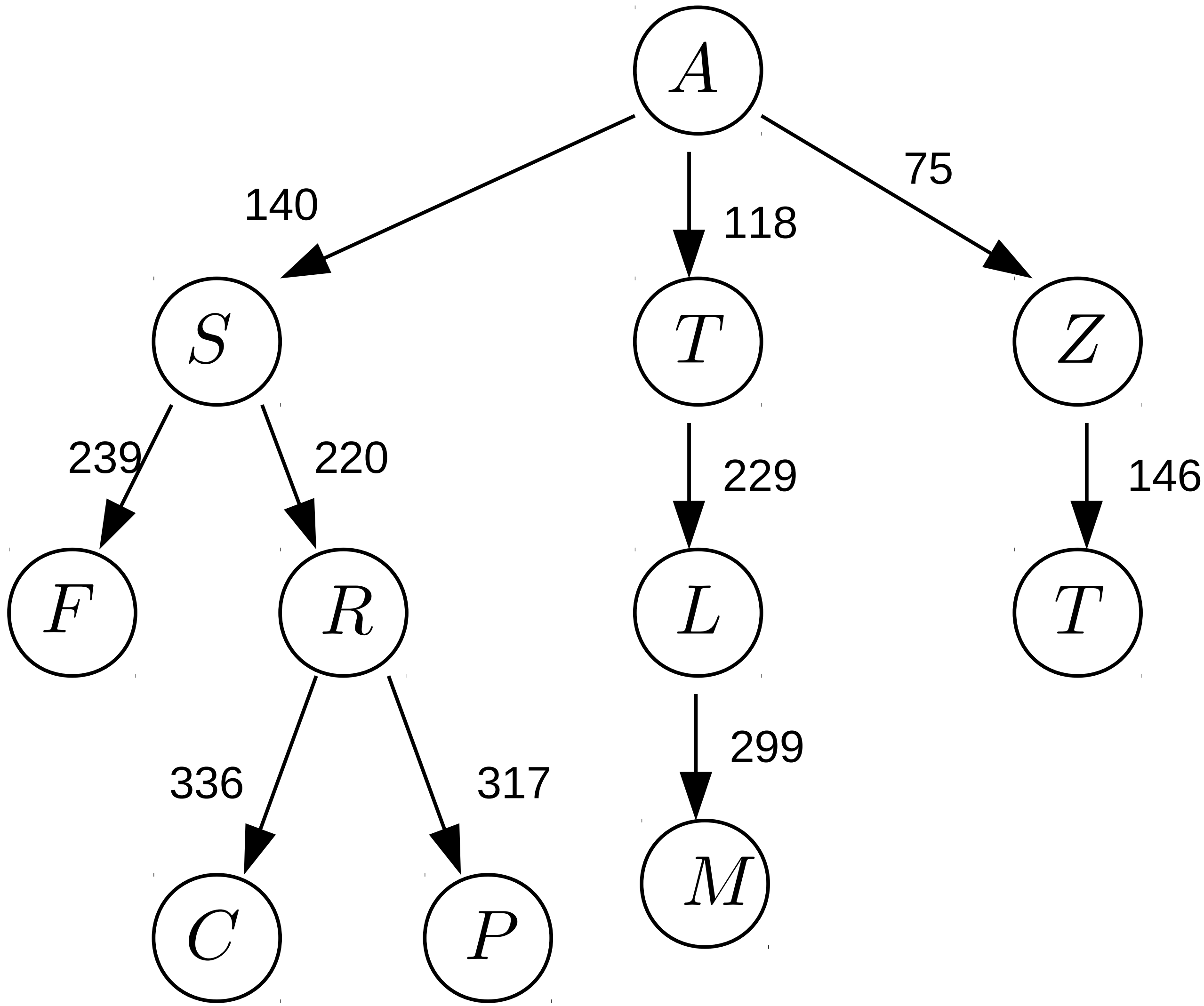
Fringe	Path Cost
~~A~~	~~0~~
~~S~~	~~140~~
~~T~~	~~118~~
~~Z~~	~~75~~
~~T~~	~~146~~
L	229
F	239
~~R~~	~~220~~
C	336
P	317



Explored set: A, Z, T, S, R

UCS

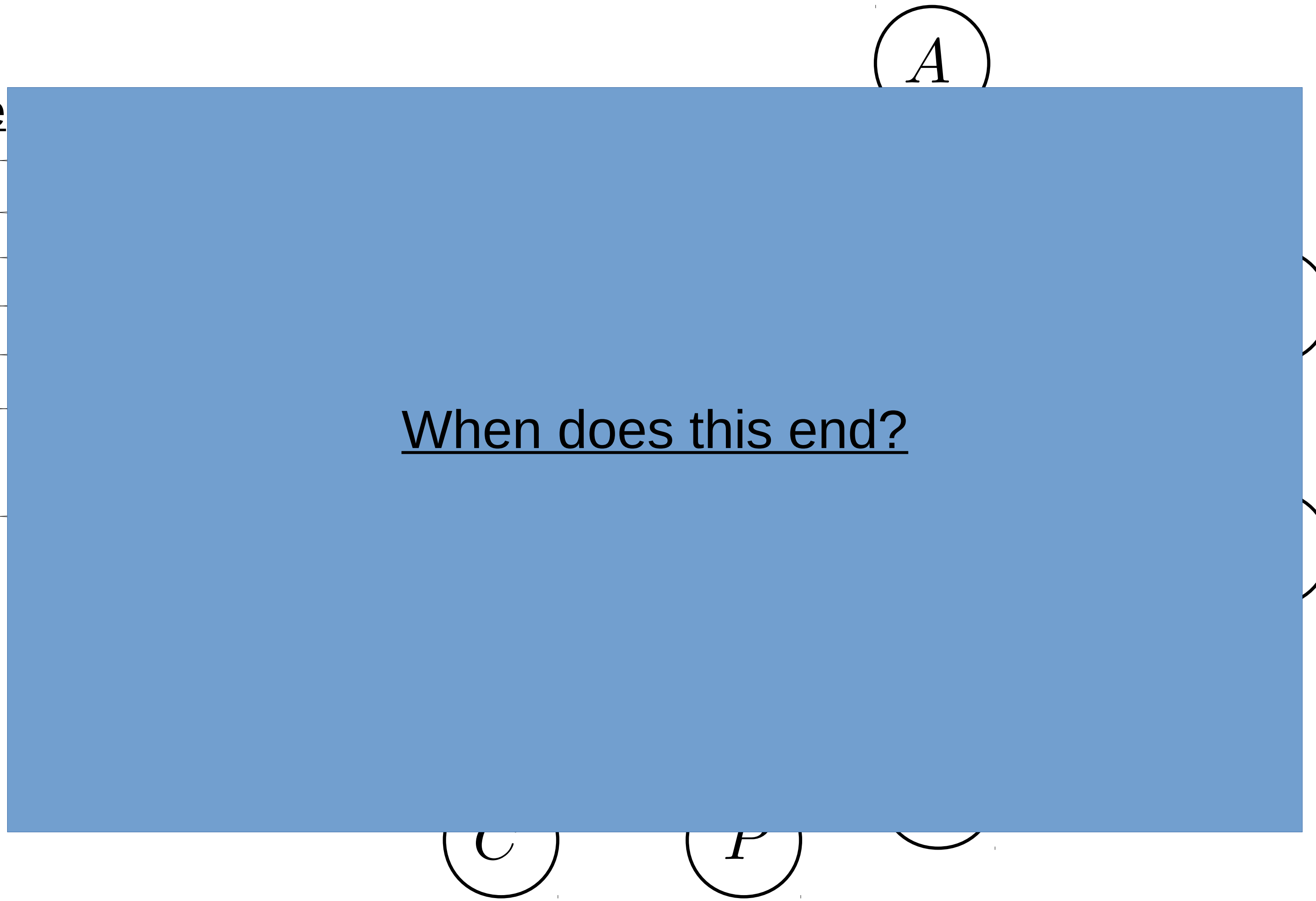
Fringe	Path Cost
~~A~~	~~0~~
~~S~~	~~140~~
~~T~~	~~118~~
~~Z~~	~~75~~
~~T~~	~~146~~
~~L~~	~~229~~
F	239
~~R~~	~~220~~
C	336
P	317
M	299



Explored set: A, Z, T, S, R, L

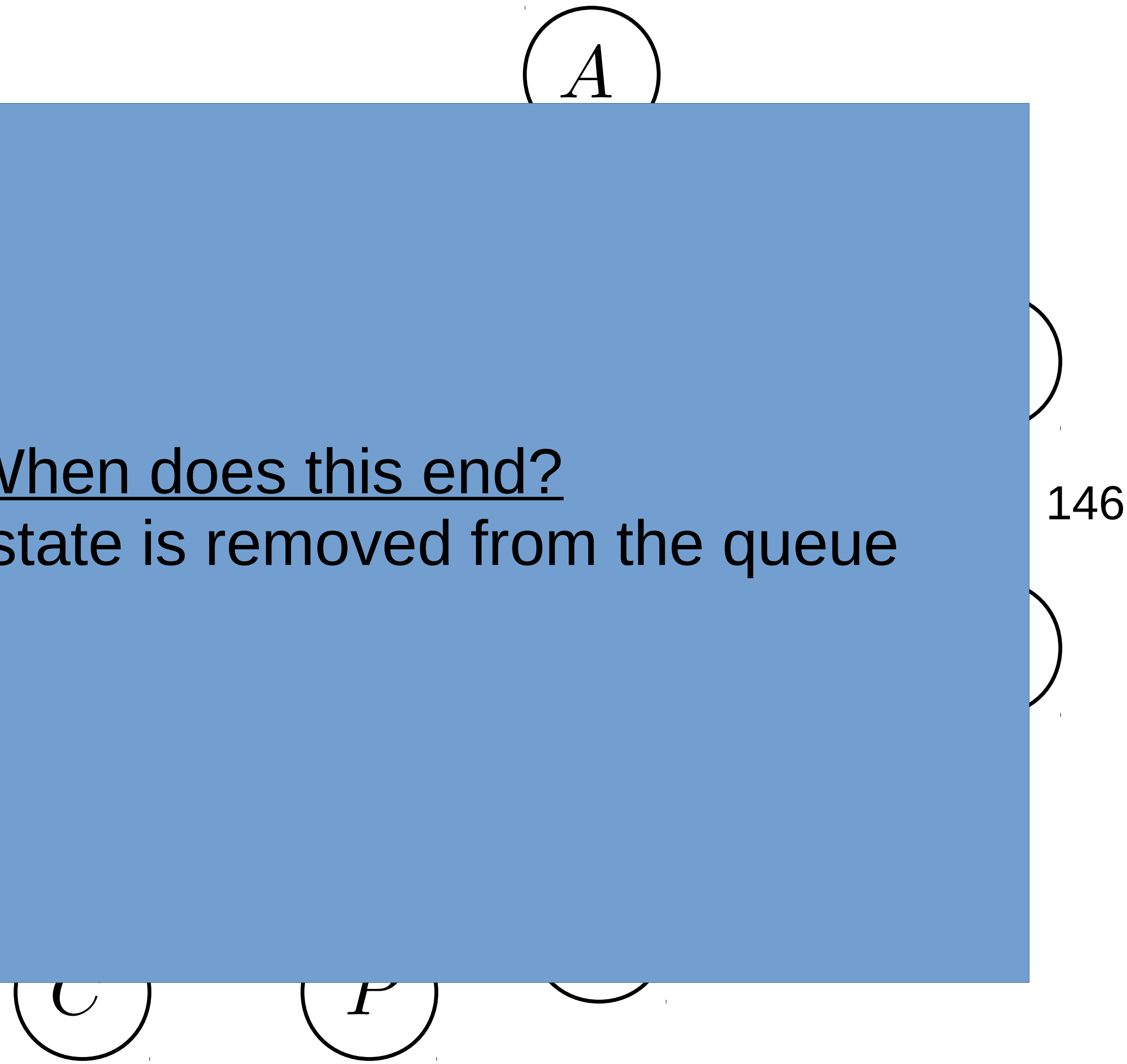
UCS

Fringe
~~A~~
~~S~~
~~T~~
~~Z~~
~~T~~
~~L~~
F
~~R~~
C
P
M



Explored set: A, Z, T, S, R, L

UCS



<u>Fringe</u>
~~A~~
~~S~~
~~T~~
~~Z~~
~~T~~
~~L~~
F
~~R~~
C
P
M

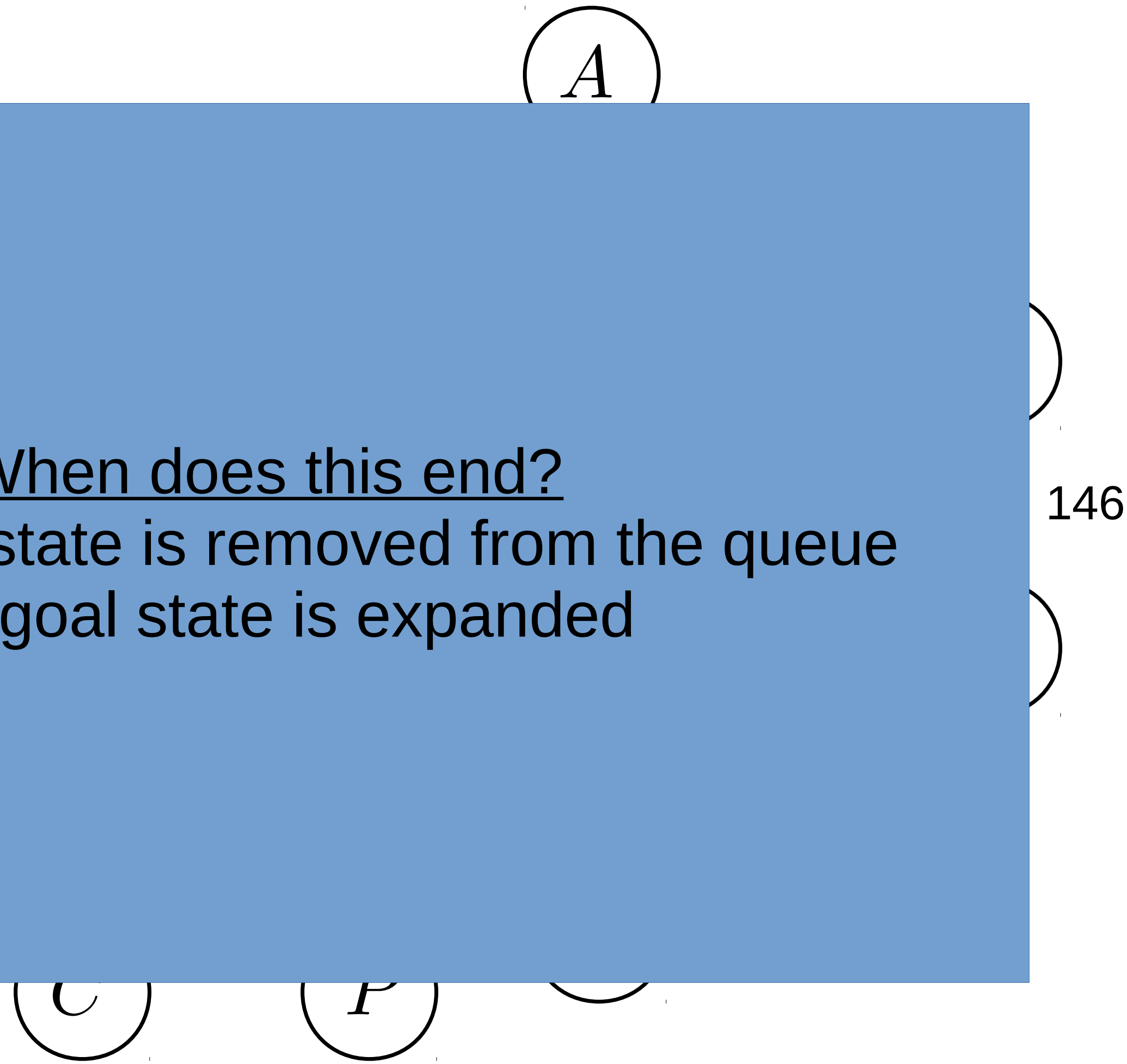
<u>When does this end?</u>
– when the goal state is removed from the queue

Explored set: A, Z, T, S, R, L

UCS

<u>Fringe</u>

~~A~~
~~S~~
~~T~~
~~Z~~
~~T~~
~~L~~
F
~~R~~
C
P
M



<u>When does this end?</u>

– when the goal state is removed from the queue
– NOT when the goal state is expanded

Explored set: A, Z, T, S, R, L

UCS

```
function UNIFORM-COST-SEARCH(problem) returns a solution, or failure
  node ← a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
  frontier ← a priority queue ordered by PATH-COST, with node as the only element
  explored ← an empty set
  loop do
     if EMPTY?(frontier) then return failure
     node ← POP(frontier)   /* chooses the lowest-cost node in frontier */
     if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
     add node.STATE to explored
     for each action in problem.ACTIONS(node.STATE) do
        child ← CHILD-NODE(problem, node, action)
        if child.STATE is not in explored or frontier then
           frontier ← INSERT(child, frontier)
        else if child.STATE is in frontier with higher PATH-COST then
           replace that frontier node with child
```

Figure 3.14 Uniform-cost search on a graph. The algorithm is identical to the general graph search algorithm in Figure 3.7, except for the use of a priority queue and the addition of an extra check in case a shorter path to a frontier state is discovered. The data structure for *frontier* needs to support efficient membership testing, so it should combine the capabilities of a priority queue and a hash table.

UCS Properties

Is UCS <u>complete</u>?

– is it guaranteed to find a solution if one exists?

What is the <u>time complexity</u> of UCS?

– how many states are expanded before finding a solution?

- – b: branching factor
- – C*: cost of optimal solution
- – e: min one-step cost
- – complexity = $O(b^{C^*/e})$

What is the <u>space complexity</u> of BFS?

– how much memory is required?

- – complexity = $O(b^{C^*/e})$

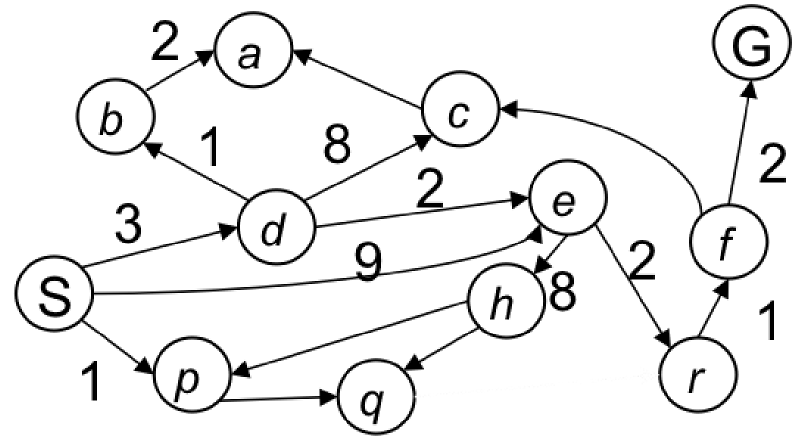
Is BFS optimal?

– is it guaranteed to find the best solution (shortest path)?

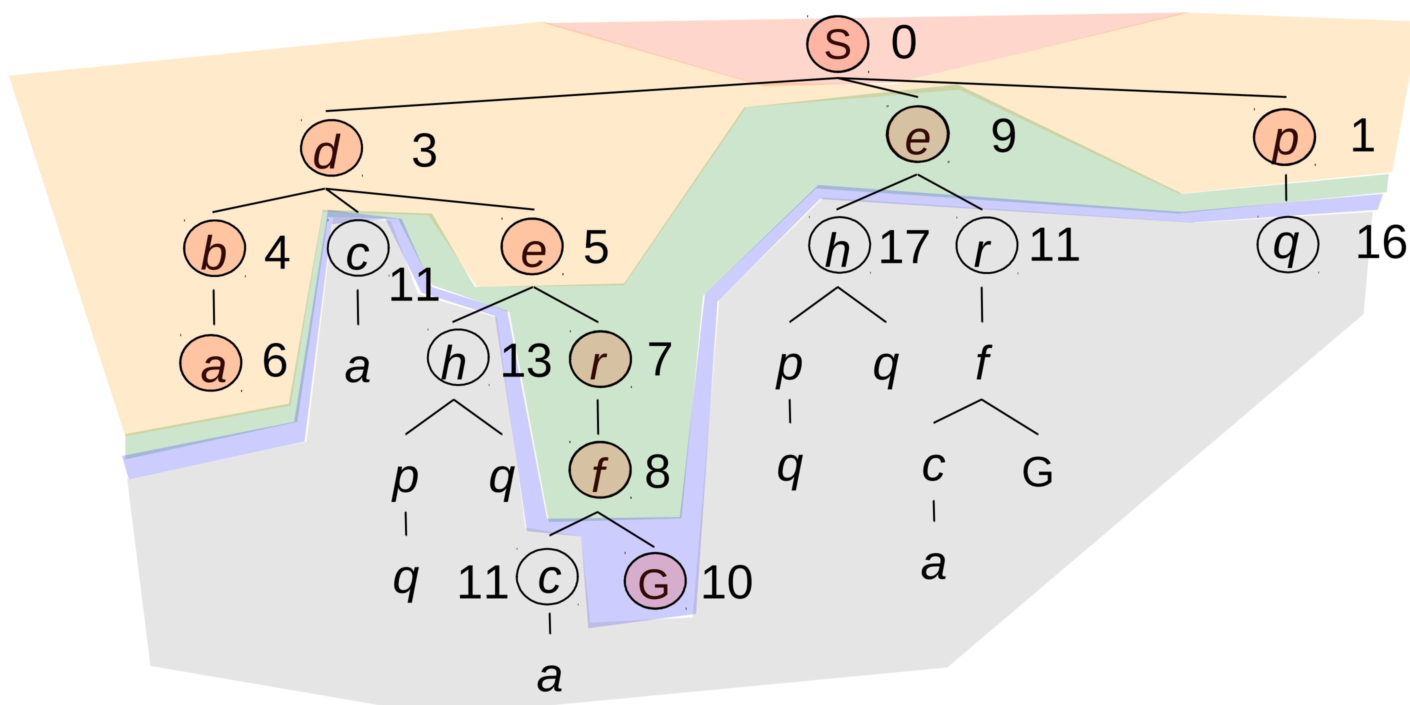
UCS vs BFS

Strategy: expand cheapest node first:

Fringe is a priority queue (priority: cumulative cost)



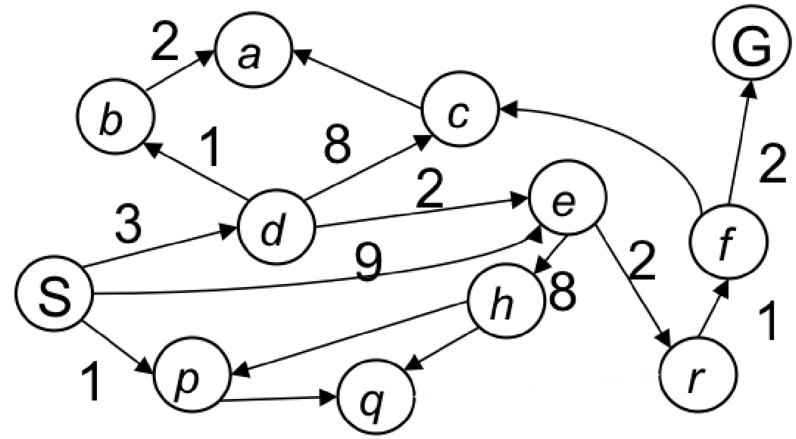
Cost contours

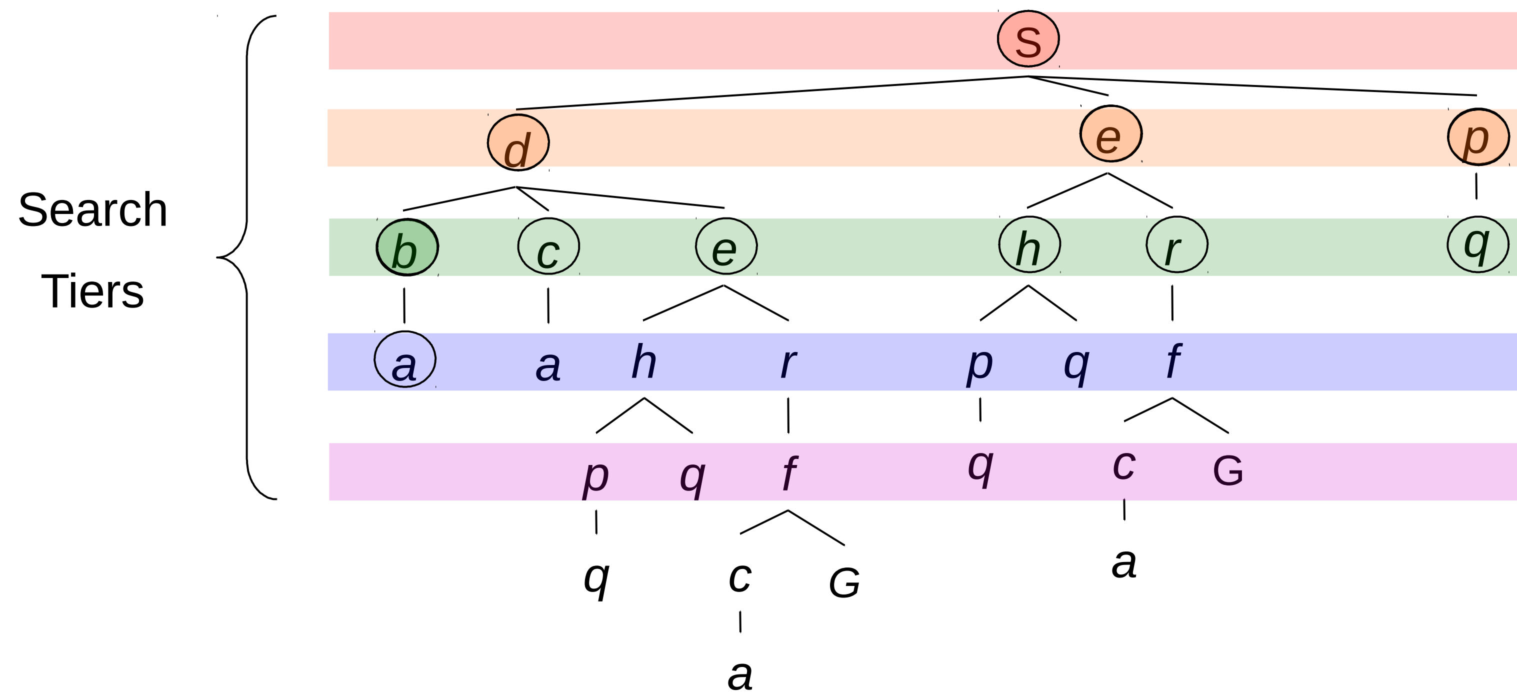


UCS vs BFS

Strategy: expand a shallowest node first

Implementation: Fringe is a FIFO queue





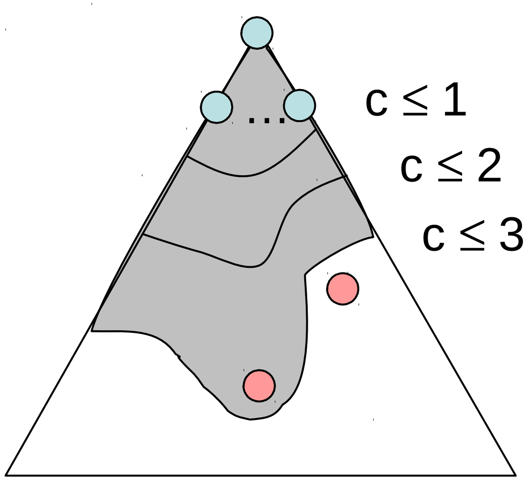
UCS vs BFS

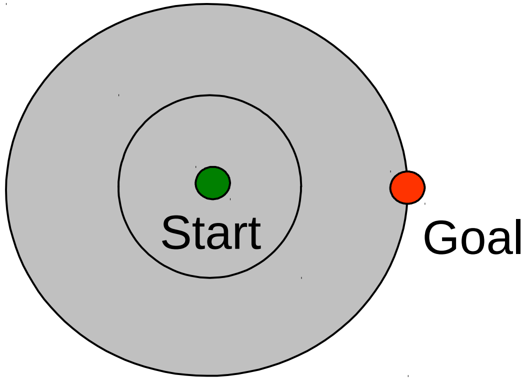
- Remember: UCS explores increasing cost contours

- The good: UCS is complete and optimal!

- The bad:
  - Explores options in every “direction”
  - No information about goal location

- We’ll fix that soon!





Depth First Search (DFS)

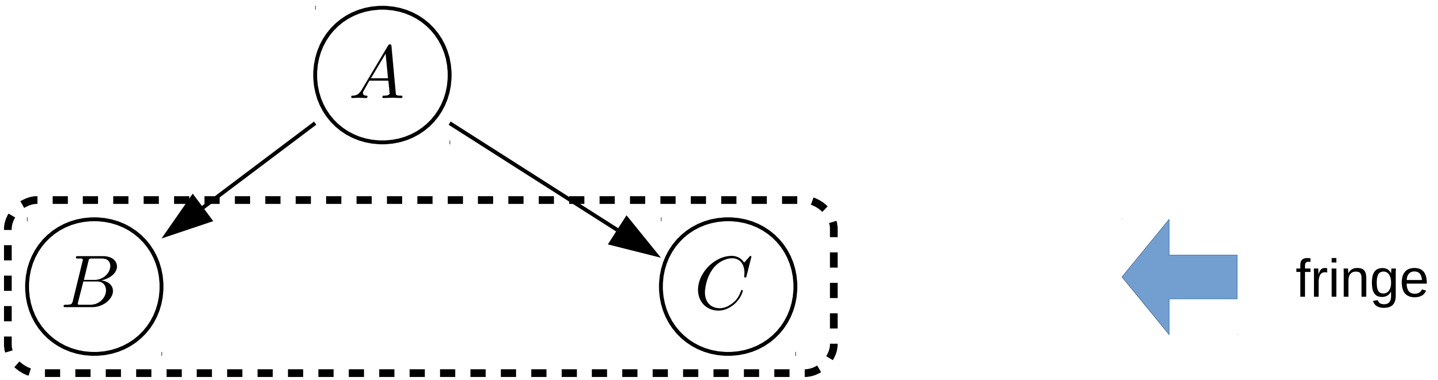
DFS

Fringe
A

DFS

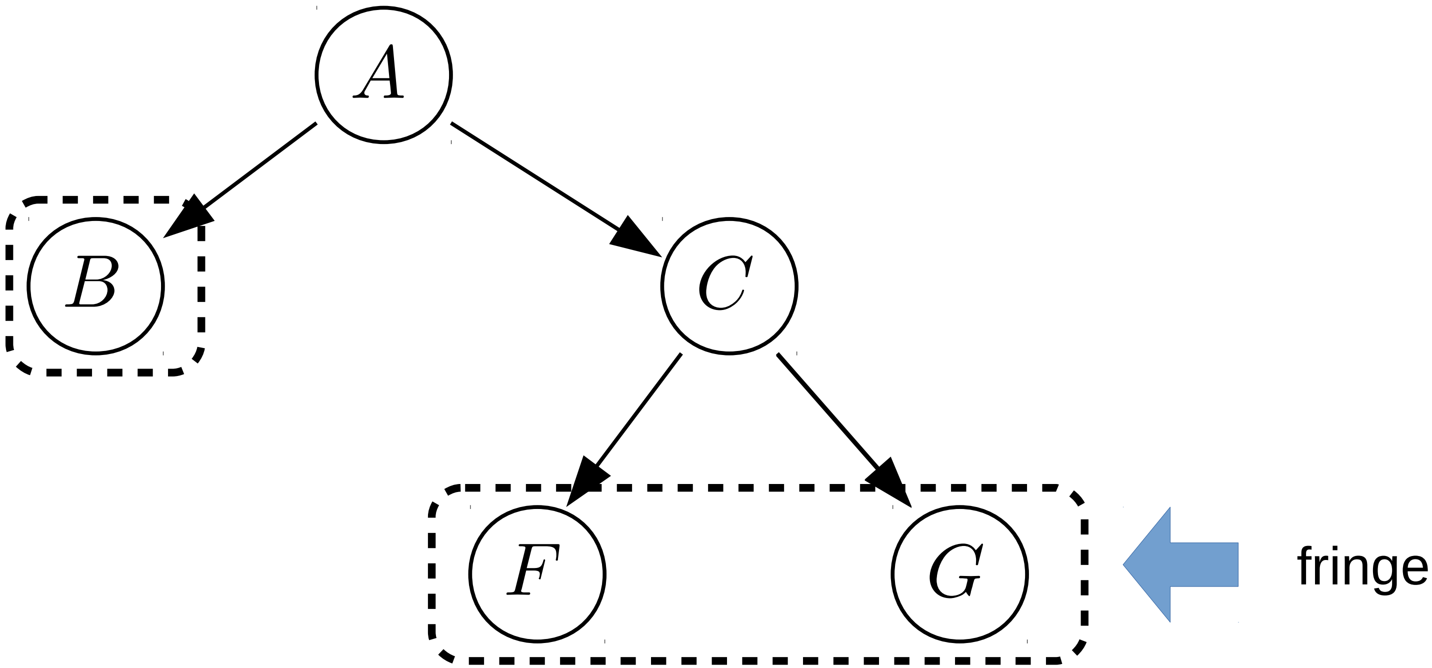
Fringe

~~A~~
B
C



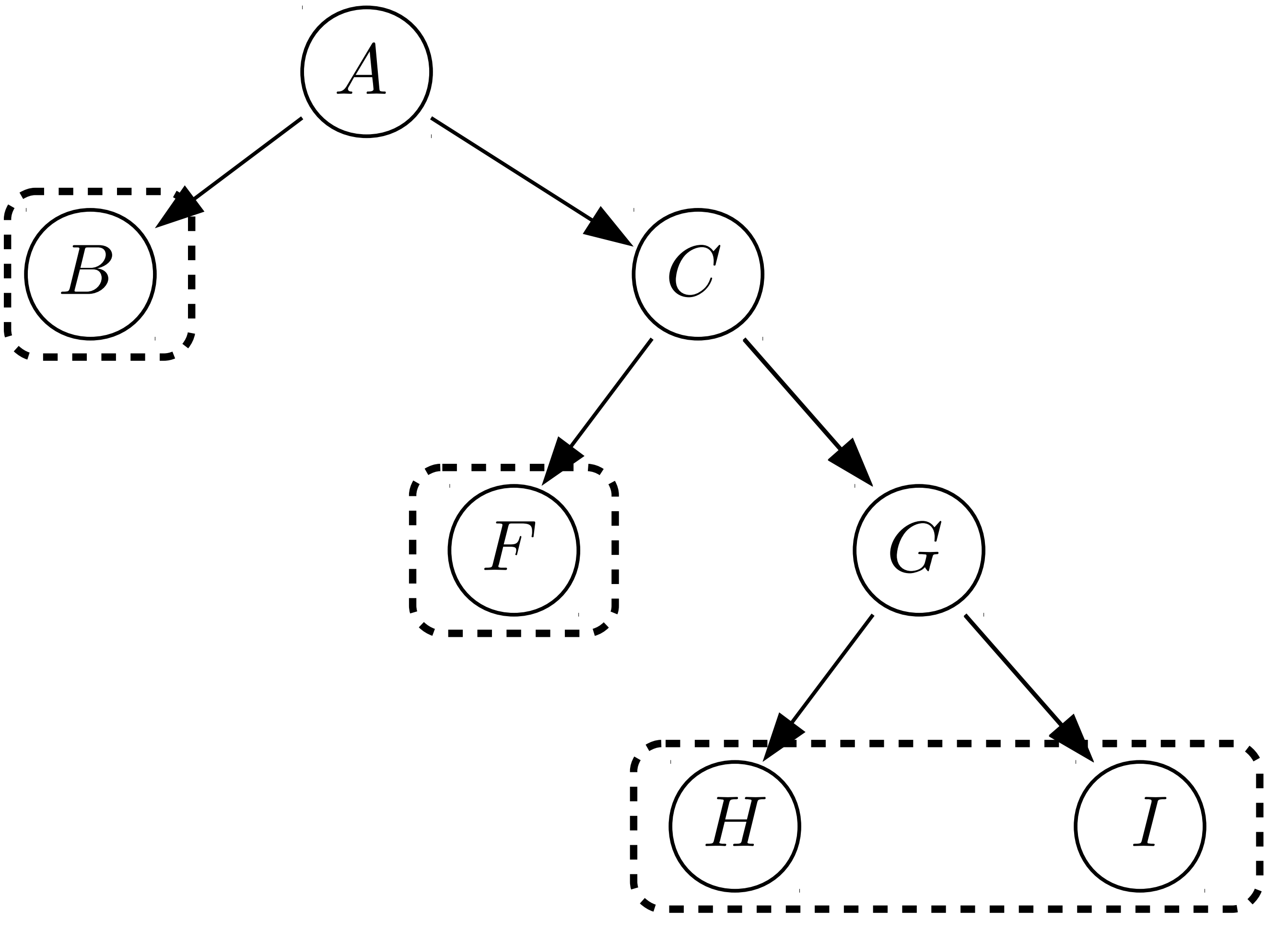
DFS

<u>Fringe</u>
~~A~~
B
~~C~~
F
G



DFS

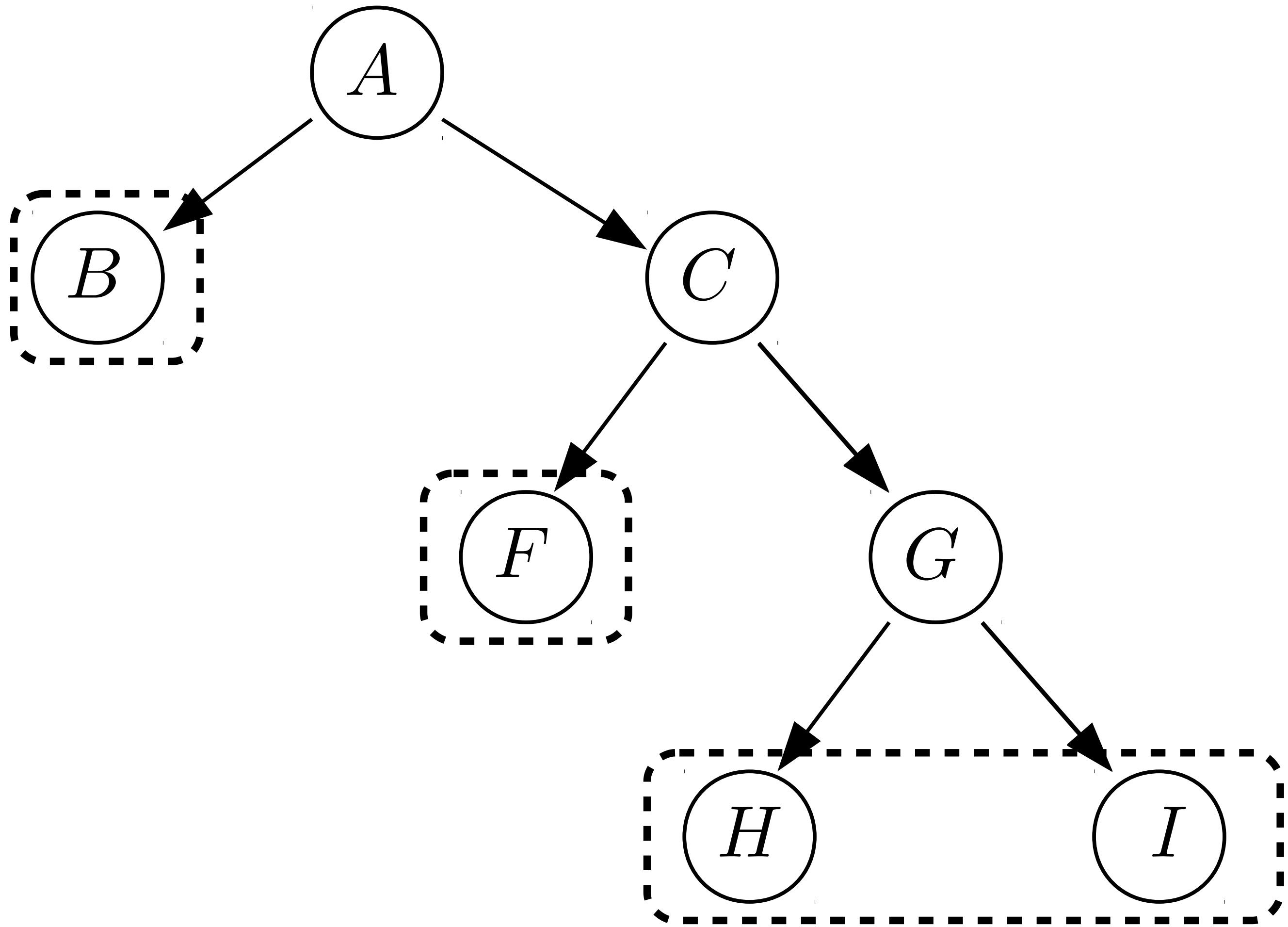
<u>Fringe</u>
~~A~~
B
~~C~~
F
~~G~~
H
I



DFS

Fringe

~~A~~
B
~~C~~
F
~~G~~
H
I

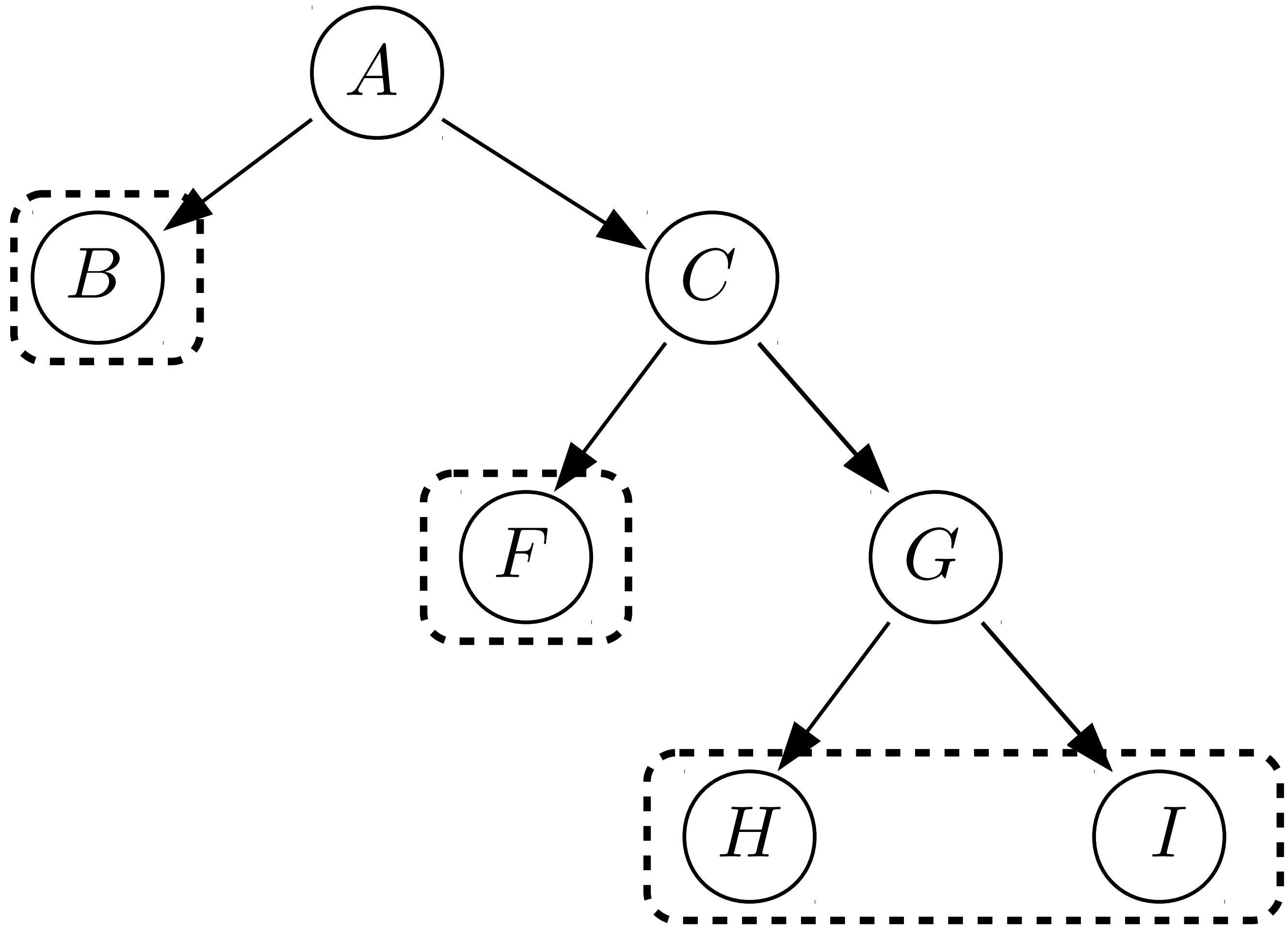


Which state gets removed next from the fringe?

DFS

<u>Fringe</u>

~~A~~
B
~~C~~
F
~~G~~
H
I

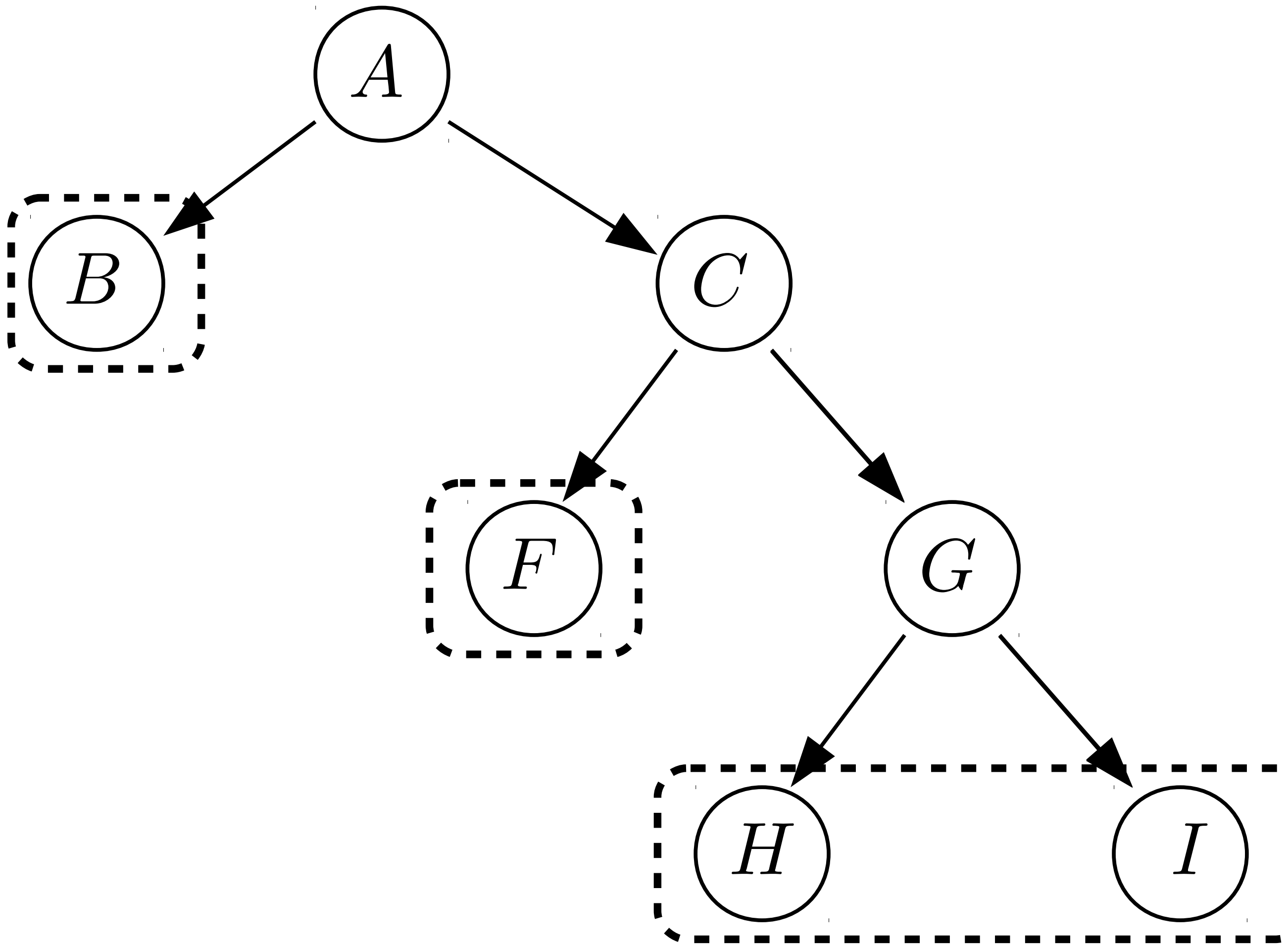


Which state gets removed next from the fringe?

What kind of a queue is this?

DFS

Fringe
~~A~~
B
~~C~~
F
~~G~~
H
I

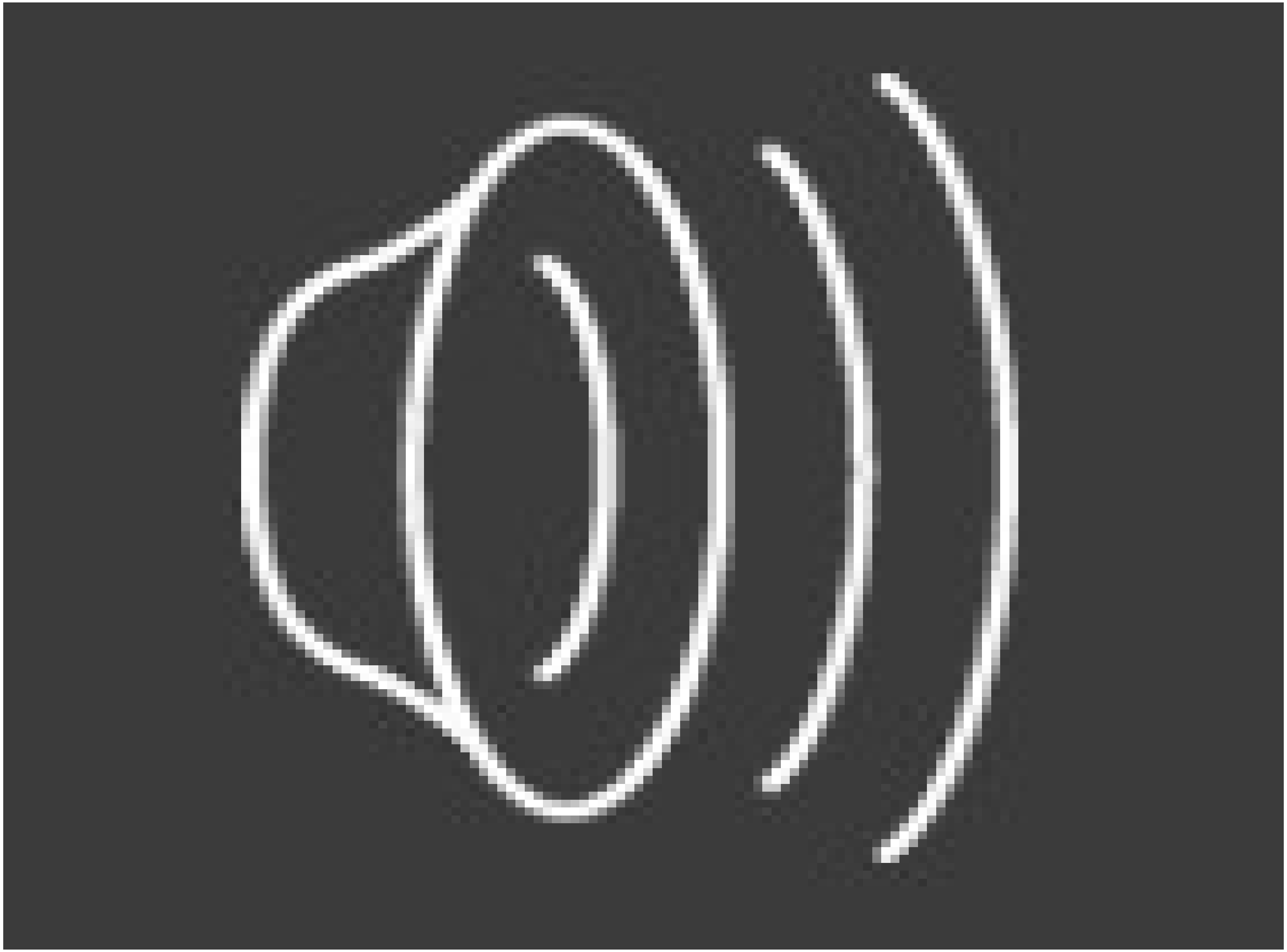


Which state gets removed next from the fringe?

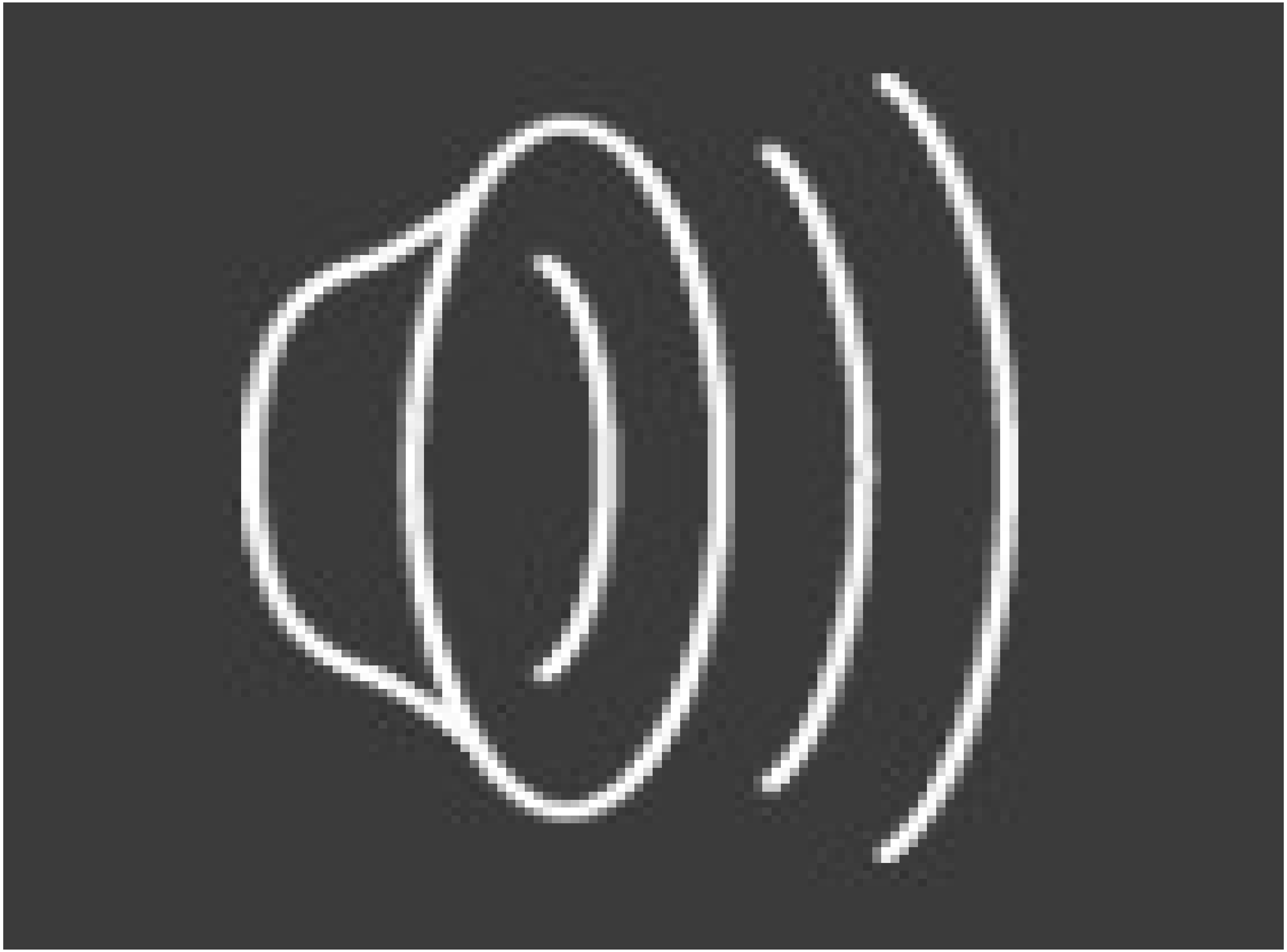
What kind of a queue is this?

LIFO Queue!
(last in first out)

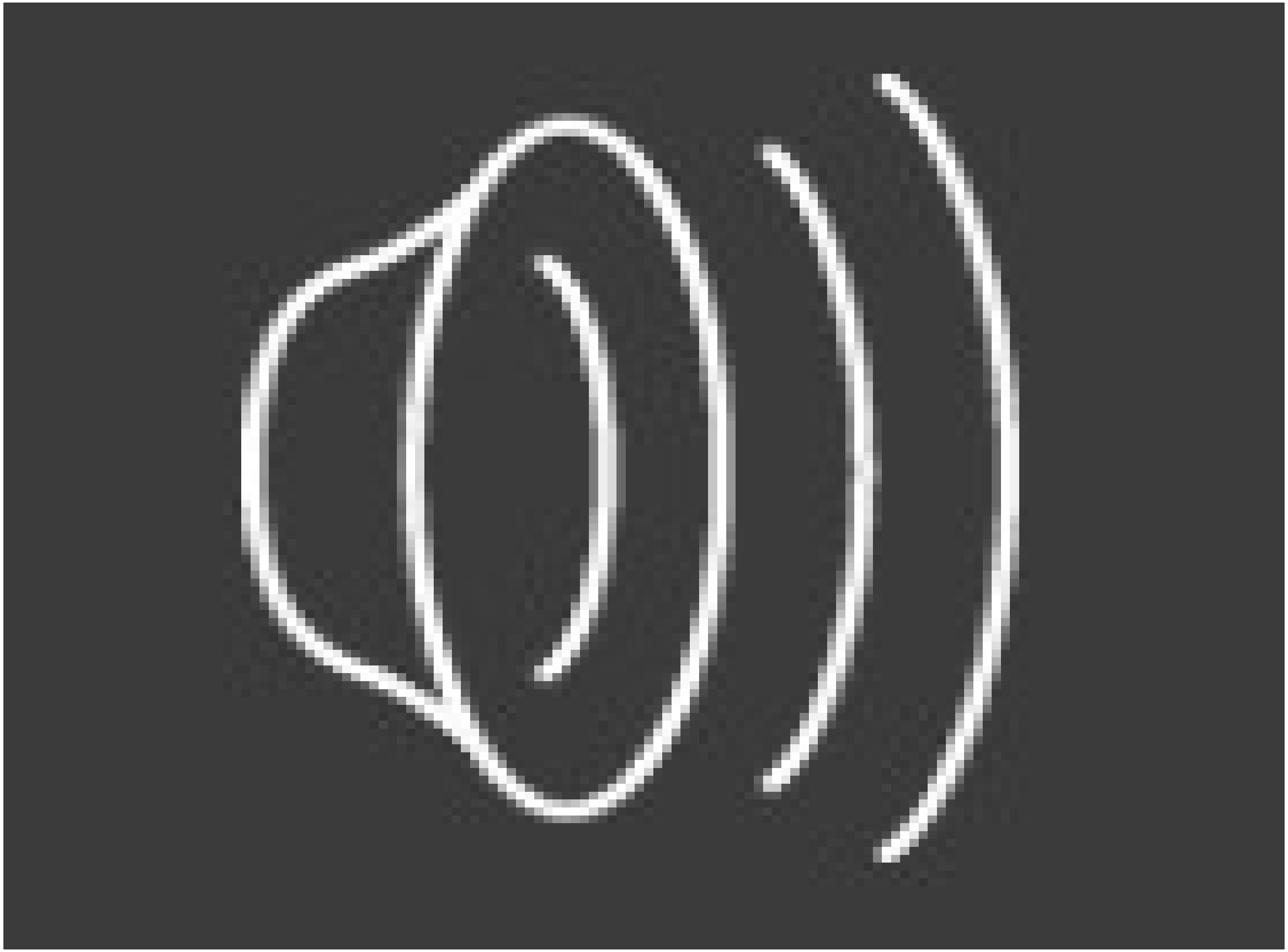
DFS vs BFS: which one is this?



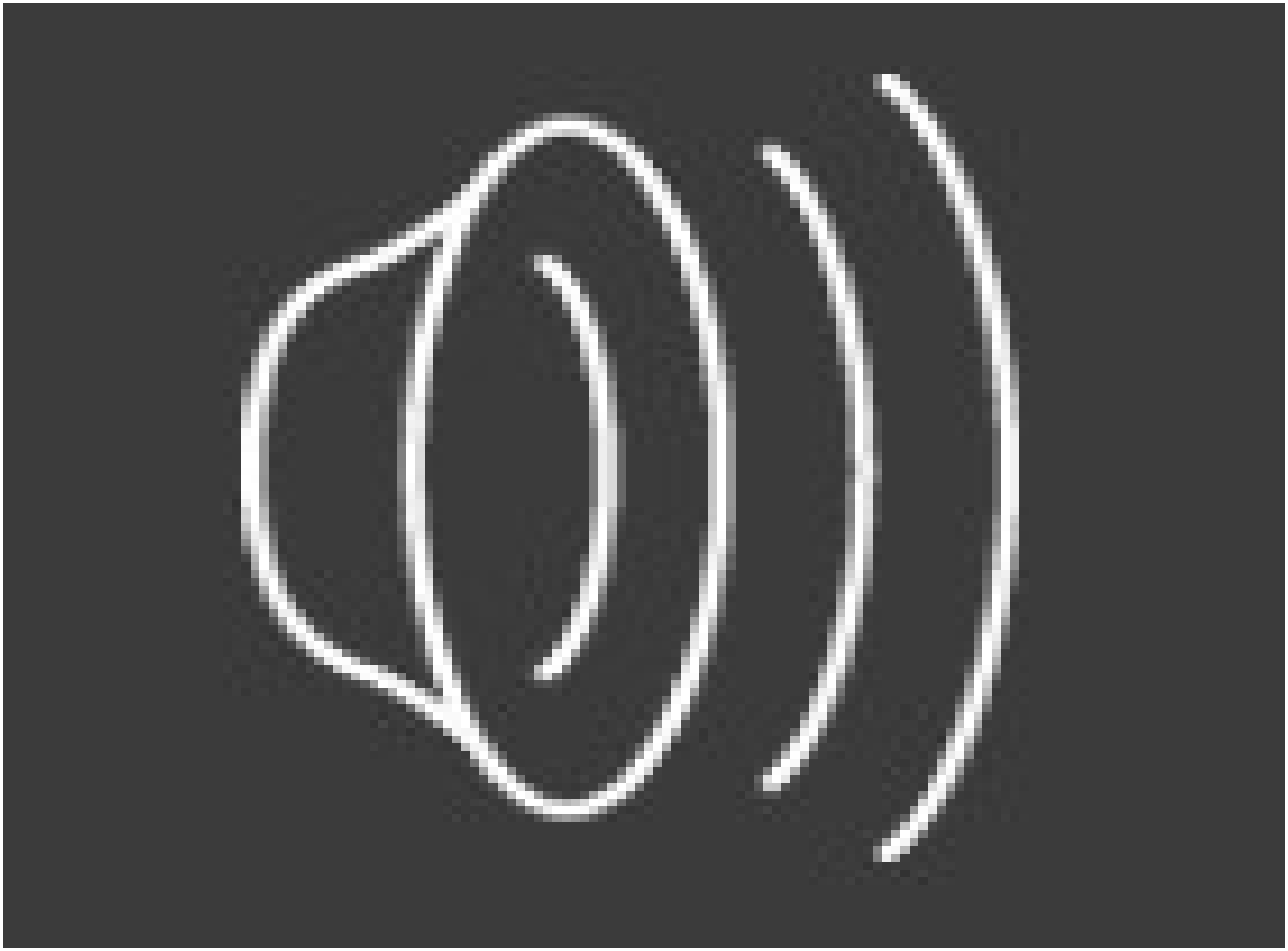
DFS vs BFS: which one is this?



BFS/UCS: which is this?



BFS/UCS: which is this?



DFS Properties: Graph search version

This is the "graph search" version of the algorithm

Is DFS <u>complete</u>?
– only if you track the explored set in memory

What is the <u>time complexity</u> of DFS (graph version)?
– how many states are expanded before finding a solution?
    – complexity = number of states in the graph

What is the <u>space complexity</u> of DFS (graph version)?
– how much memory is required?
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Is DFS optimal?
– is it guaranteed to find the best solution (shortest path)?

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Is DFS optimal?
– is it guaranteed to find the best solution (shortest path)?

So why would we ever use this algorithm?

DFS: Tree search version

This is the “tree search” version of the algorithm

Suppose you don't track the explored set.
– why wouldn't you want to do that?

DFS: Tree search version

This is the “tree search” version of the algorithm

Suppose you don't track the explored set.
– why wouldn't you want to do that?

What is the space complexity of DFS (tree version)?
– how much memory is required?
  – b: branching factor
  – m: maximum depth of any node
  – complexity = $O(bm)$

DFS: Tree search version

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Suppose you don't track the explored set.
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What is the space complexity of DFS (tree version)?
– how much memory is required?
  – b: branching factor
  – m: maximum depth of any node
  – complexity = $\boxed{O(bm)}$

This is why we might want to use DFS

DFS: Tree search version

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What is the space complexity of DFS (tree version)?
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  – complexity =

DFS: Tree search version

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Is it complete?

DFS: Tree search version

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What is the time complexity of DFS (tree version)?
– how many states are expanded before finding a solution? $O(b^m)$
  – complexity

Is it complete? NO!

DFS: Tree search version

This is the "tree search" version of the algorithm

Suppose you don't track the explored set.
– why wouldn't you want to do that?

What is the space complexity of DFS (tree version)?
– how much memory is required?
  – b: branching factor
  – m: maximum depth of any node
  – complexity = $O(bm)$

What is the time complexity of DFS (tree version)?
– how many states are expanded before finding a solution?
  – complexity = $O(b^m)$

Is it complete?

NO!
What do we do???

IDS: Iterative deepening search

What is IDS?
– do depth-limited DFS in stages, increasing the maximum depth at each stage

IDS: Iterative deepening search

What is IDS?
– do depth-limited DFS in stages, increasing the maximum depth at each stage

What is depth limited search?
– any guesses?

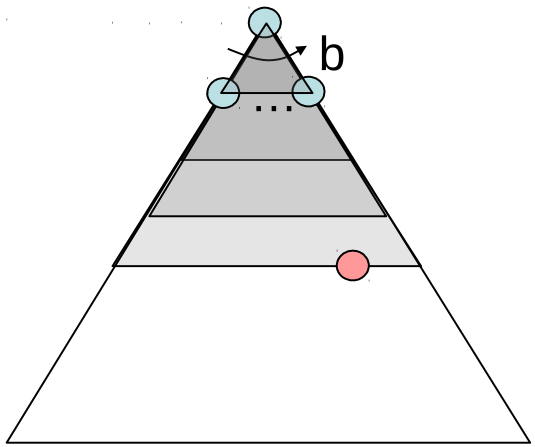
IDS: Iterative deepening search

What is IDS?
– do depth-limited DFS in stages, increasing the maximum depth at each stage

What is depth limited search?
– do DFS up to a certain pre-specified depth

IDS: Iterative deepening search

- Idea: get DFS's space advantage with BFS's time / shallow-solution advantages
  - Run a DFS with depth limit 1. If no solution…
  - Run a DFS with depth limit 2. If no solution…
  - Run a DFS with depth limit 3. …..



- Isn't that wastefully redundant?
  - Generally most work happens in the lowest level searched, so not so bad!

IDS

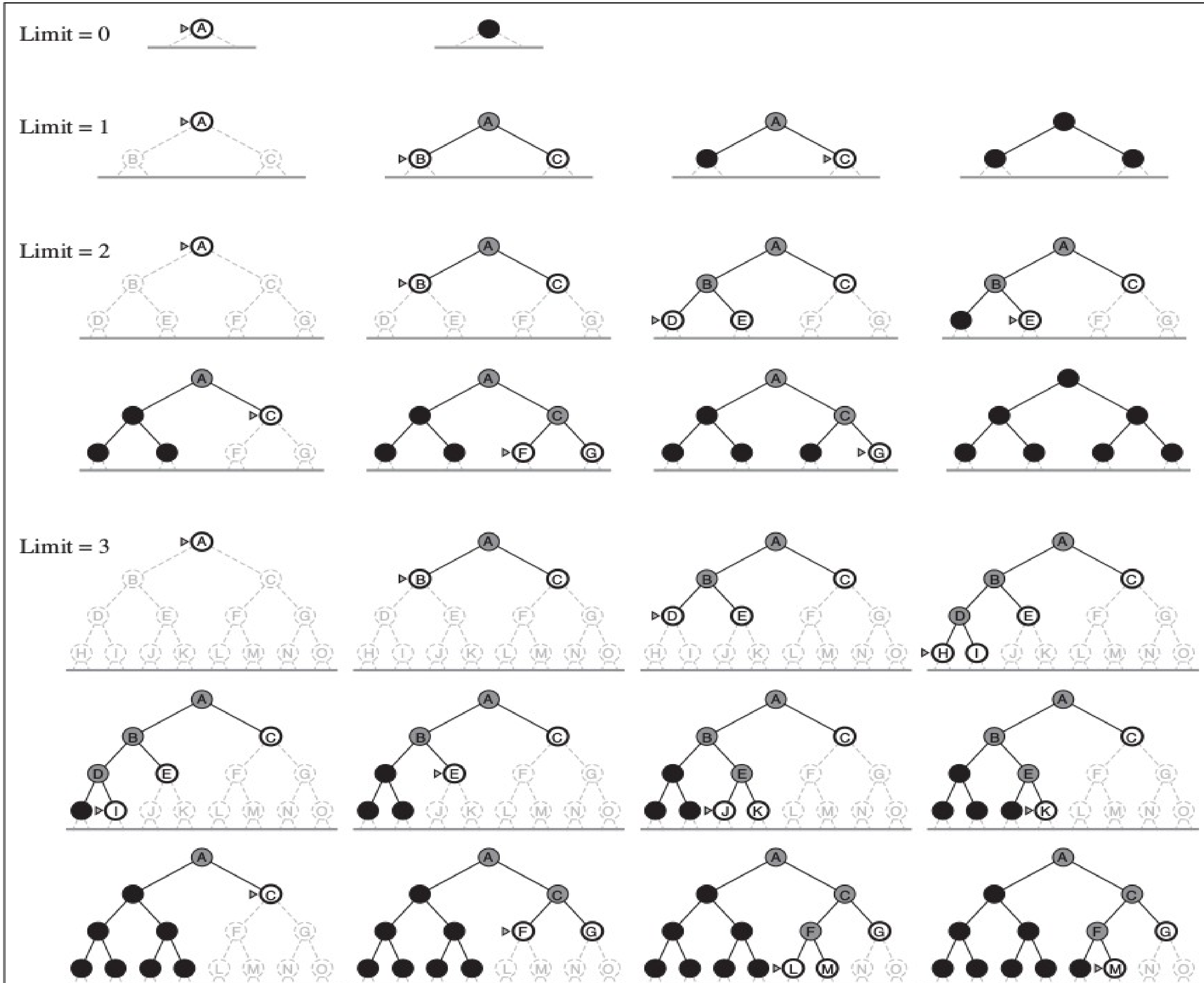


Figure 3.19 Four iterations of iterative deepening search on a binary tree.

IDS

What is the <u>space complexity</u> of IDS (tree version)?
– how much memory is required?

- b: branching factor
- m: maximum depth of any node
- complexity = $O(bm)$

What is the <u>time complexity</u> of DFS (tree version)?
– how many states are expanded before finding a solution? $O(b^m)$

- complexity =

Is it complete?

IDS

What is the <u>space complexity</u> of IDS (tree version)?
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What is the <u>time complexity</u> of DFS (tree version)?
– how many states are expanded before finding a solution? $O(b^m)$
  – complexity =

Is it complete? YES!!!

Is it optimal?

IDS

What is the <u>space complexity</u> of IDS (tree version)?
– how much memory is required?

- b: branching factor
- m: maximum depth of any node
- complexity = $O(bm)$

What is the <u>time complexity</u> of DFS (tree version)?
– how many states are expanded before finding a solution?

- complexity = $O(b^m)$

Is it complete? YES!!!

Is it optimal? YES!!!

General thoughts about search

If your model is wrong, then your solution will be wrong.

- In November 2010, Nicaraguan troops unknowingly crossed the border to Costa Rica, removed that country's flag and replaced it with their own. The reason: Google Maps told the troops' commander the territory belonged to Nicaragua.