Basic Probability

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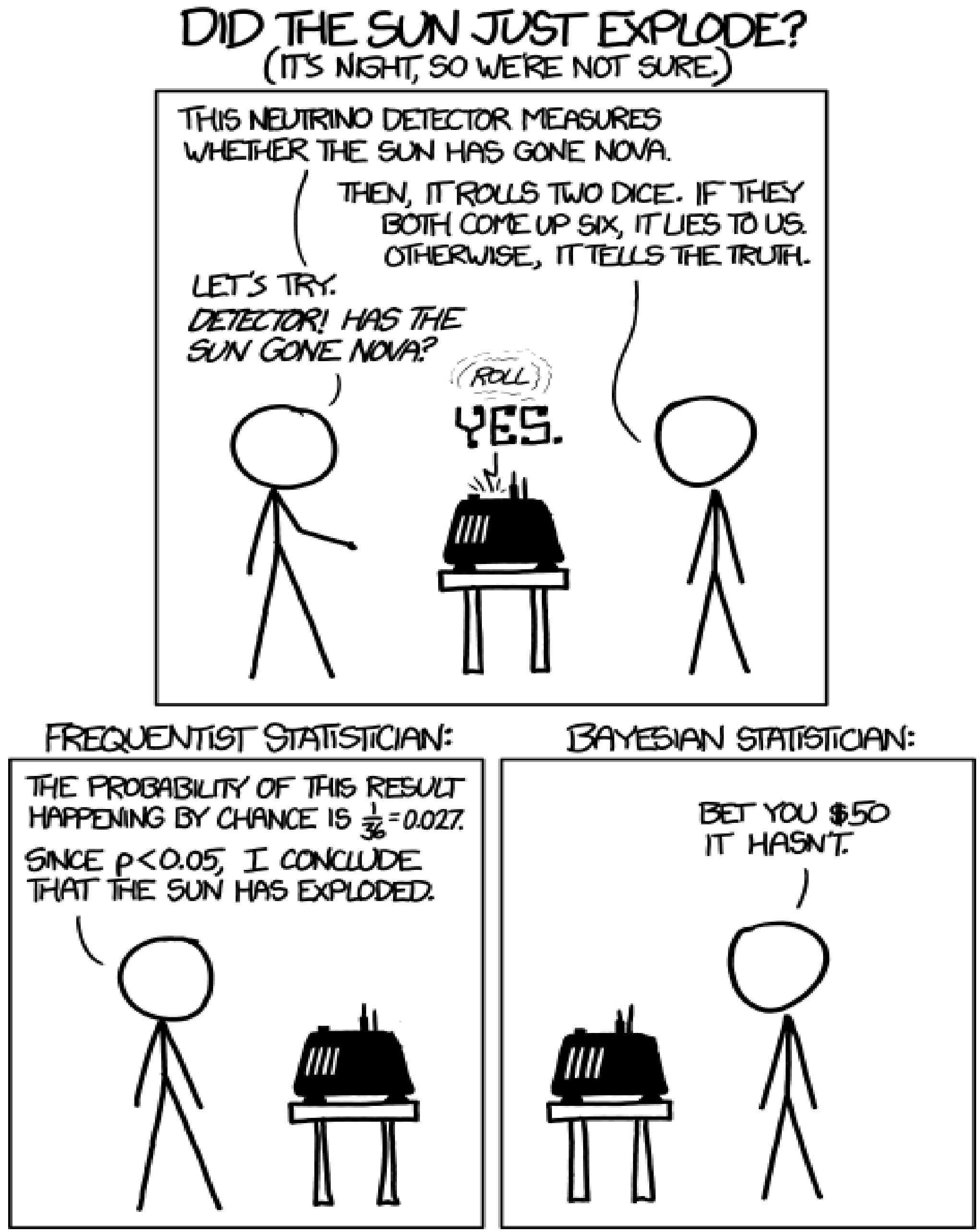
Some images and slides are used from:
1. CS188 UC Berkeley
2. RN, AIMA

Definition

- Probability theory is nothing but common sense reduced to calculation. ~Pierre Laplace

- What is probability? What does it mean when we say “the probability that a coin will land head is 0.5”

Frequentist Vs Bayesian



Random variables

What is a random variable?

Suppose that the variable *a* denotes the outcome of a role of a single six-sided die:

$$a \in \{1, 2, 3, 4, 5, 6\} = A$$

a is a *random variable*

this is the *domain* of *a*

Another example:

Suppose *b* denotes whether it is raining or clear outside:

$$b \in \{rain, clear\} = B$$

Probability distribution

A probability distribution associates each with a probability of occurrence.

A probability table is one way to encode the distribution:

$$a \in \{1,2,3,4,5,6\} = A \qquad b \in \{rain, clear\} = B$$

a	P(a)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b	P(b)
rain	1/4
clear	3/4

<u>All probability distributions must satisfy the following:</u>

1. $\forall a \in A, a \geq 0$
2. $\sum_{a \in A}, a = 1$

Writing probabilities

a	P(a)
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

b	P(b)
rain	1/4
clear	3/4

For example: $p(a = 2) = 1/6$

$$p(b = clear) = 3/4$$

But, sometimes we will abbreviate this as: $p(2) = 1/6$

$$p(clear) = 3/4$$

Joint probability distributions

Given random variables: $X_1, X_2, \ldots, X_n$

The *joint distribution* is a probability assignment to all combinations:

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

or:

$$P(x_1, x_2, \ldots, x_n)$$

As with single-variate distributions, joint distributions must satisfy:

1. $$P(x_1, x_2, \ldots, x_n) \geq 0$$

2. $$\sum_{x_1, \ldots, x_n} P(x_1, x_2, \ldots, x_n) = 1$$

Joint probability distributions

Joint distributions are typically written in table form:

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Marginalization

Given P(T,W), calculate P(T) or P(W)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(T) = \sum_{w \in W} P(T, w)$$

T	P(T)
hot	0.5
cold	0.5

$$P(W) = \sum_{t \in T} P(t, W)$$

W	P(W)
sun	0.5
rain	0.4

Marginalization

$P(X,Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

$$P(x) = \sum_y P(x,y)$$

$$P(y) = \sum_x P(x,y)$$

$P(X)$

X	P
+x	
-x	

$P(Y)$

Y	P
+y	
-y	

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Conditional Probabilities

$P(sun|hot) \equiv$ Probability that it is sunny *given* that it is hot.

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Probabilities

Calculate the conditional probability using the product rule:

Product rule

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s,T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s,T=c) + P(W=r,T=c)$$

$$= 0.2 + 0.3 = 0.5$$

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Conditional Probabilities

- P(+x | +y) ?

$P(X, Y)$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

- P(-x | +y) ?

- P(-y | +x) ?

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Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

W	P(W $\mid t = hot$)
sun	0.8
rain	0.2

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

W	P(W | $t = hot$)
sun	0.8
rain	0.2

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

$$P(sun|hot) = \frac{P(sun,hot)}{P(hot)} = \frac{P(sun,hot)}{P(sun,hot) + P(rain,hot)}$$

$$= \frac{0.4}{0.4 + 0.1}$$

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

W	P(W $\mid t = hot$)
sun	0.8
rain	0.2

W	P(W $\mid t = cold$)
sun	0.4
rain	0.6

Conditional distribution

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W|t) = \frac{P(W,t)}{P(t)}$$

W	P(W | $t = hot$)
sun	0.8
rain	0.2

W	P(W | $t = cold$)
sun	0.4
rain	0.6

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{P(sun, cold)}{P(sun, cold) + P(rain, cold)}$$

$$= \frac{0.2}{0.2 + 0.3}$$

Normalization

Given P(T,W), calculate P(T|w) or P(W|t)...

T	W	P(T,W)

Can we avoid explicitly computing this?

$$\ldots t) = \frac{P(W,t)}{P(t)}$$

W	P(W $\mid t = hot$)
sun	0.8
rain	0.2

W	P(W $\mid t = cold$)
sun	0.4
rain	0.6

$$P(sun|cold) = \frac{P(sun,cold)}{P(cold)} = \frac{P(sun,cold)}{P(sun,cold) + P(rain,cold)}$$

$$= \frac{0.2}{0.2 + 0.3}$$

Normalization

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

→

W	P(W, t=hot)
sun	0.4
rain	0.1

→

W	$P(W \mid t = hot)$
sun	0.8
rain	0.2

Select corresponding elts
from the joint distribution

Scale the numbers so
that they sum to 1.

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{P(sun, cold)}{P(sun, cold) + P(rain, cold)}$$

Normalization

T	W	P(T,W)
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

→

W	P(W, t=hot)
sun	0.4
rain	0.1

→

W	$P(W \mid t = hot)$
sun	0.8
rain	0.2

Select corresponding elts from the joint distribution

Scale the numbers so that they sum to 1.

$$P(sun|cold) = \frac{P(sun, cold)}{P(cold)} = \frac{P(sun, cold)}{P(sun, cold) + P(rain, cold)}$$

The only purpose of this denominator is to make the distribution sum to one.
– we achieve the same thing by scaling.

Normalization

P(X | Y=-y) ?

$$P(X, Y)$$

X	Y	P
+x	+y	0.2
+x	-y	0.3
-x	+y	0.4
-x	-y	0.1

→ ? → ?

Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

It's easy to derive from the product rule:

$$P(a,b) = P(b|a)P(a) = \underbrace{P(a|b)}P(b)$$

Solve for this

Using Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Using Bayes Rule

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

But harder to estimate this

It's often easier to estimate this

Bayes Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Suppose you have a stiff neck...

Suppose there is a 70% chance of meningitis if you have a stiff neck:

stiff neck meningitis

$$P(s|m) = 0.7$$

What are the chances that you have meningitis?

Bayes Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

Suppose you have a stiff neck...

Suppose there is a 70% chance of meningitis if you have a stiff neck:

stiff neck   meningitis

$$P(s|m) = 0.7$$

What are the chances that you have meningitis?

We need a little more information...

Bayes Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

$$P(s|m) = 0.7$$

$$P(s) = 0.01$$ ← Prior probability of stiff neck

$$P(m) = \frac{1}{50000}$$ ← Prior probability of meningitis

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times \frac{1}{50000}}{0.01} = 0.0014$$

Bayes Rule Example

$$P(cause|effect) = \frac{P(effect|cause)P(cause)}{P(effect)}$$

$$P(s|m) = 0.7$$

$$P(s) = 0.01$$

Prior probability of stiff neck

$$P(m) = \frac{1}{50000}$$

Prior probability of meningitis

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.7 \times \frac{1}{50000}}{0.01} = 0.0014$$

Bayes Rule Example

- Given:

$P(W)$

R	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

- What is P(W | dry) ?

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