Heuristic Search

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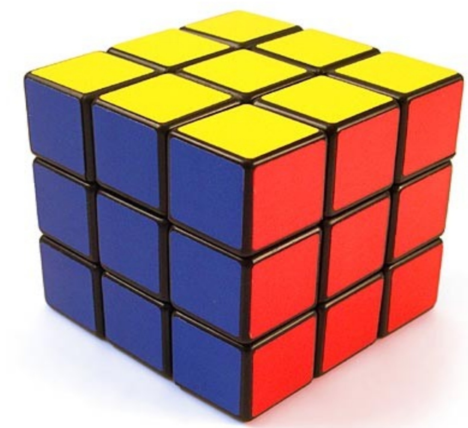
Some images and slides are used from:
1. CS188 UC Berkeley
2. RN, AIMA

Recap: What is graph search?

Start state



Goal state

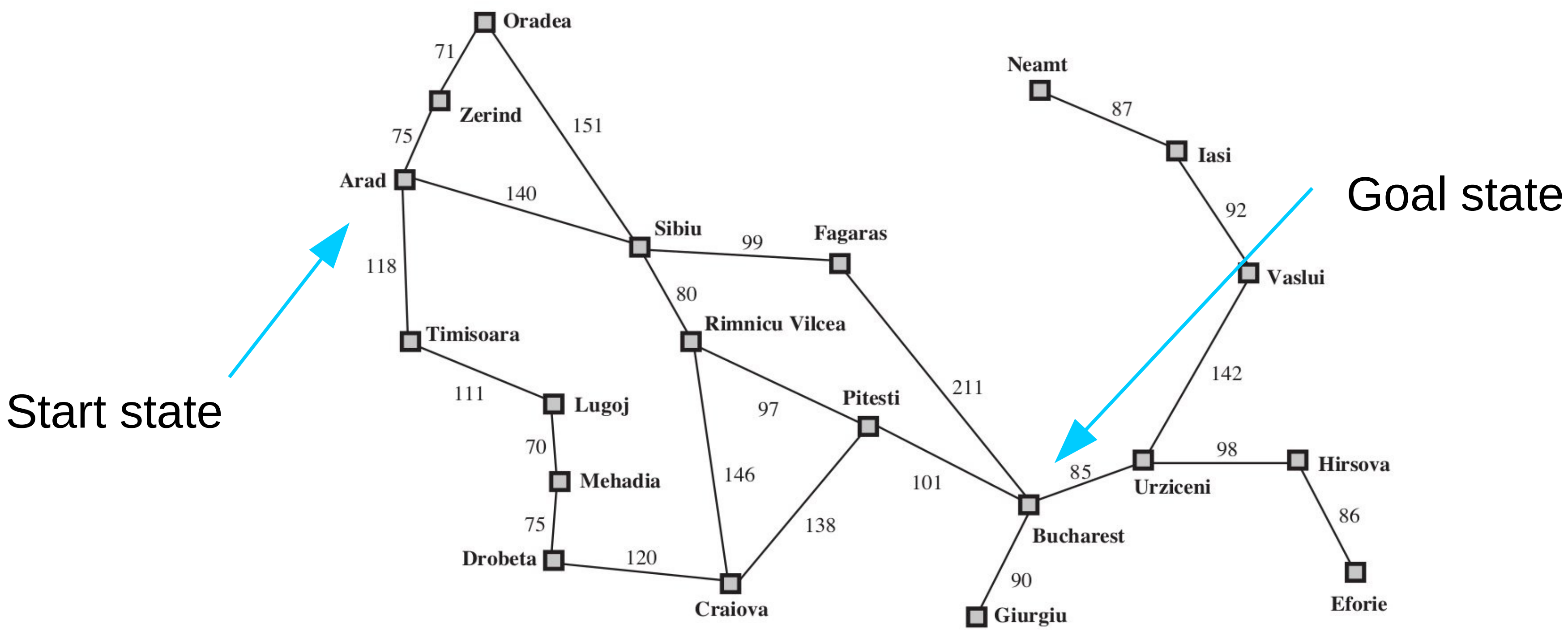
Graph search: find a path from start to goal

– what are the states?

– what are the actions (transitions)?

– how is this a graph?

Recap: What is graph search?



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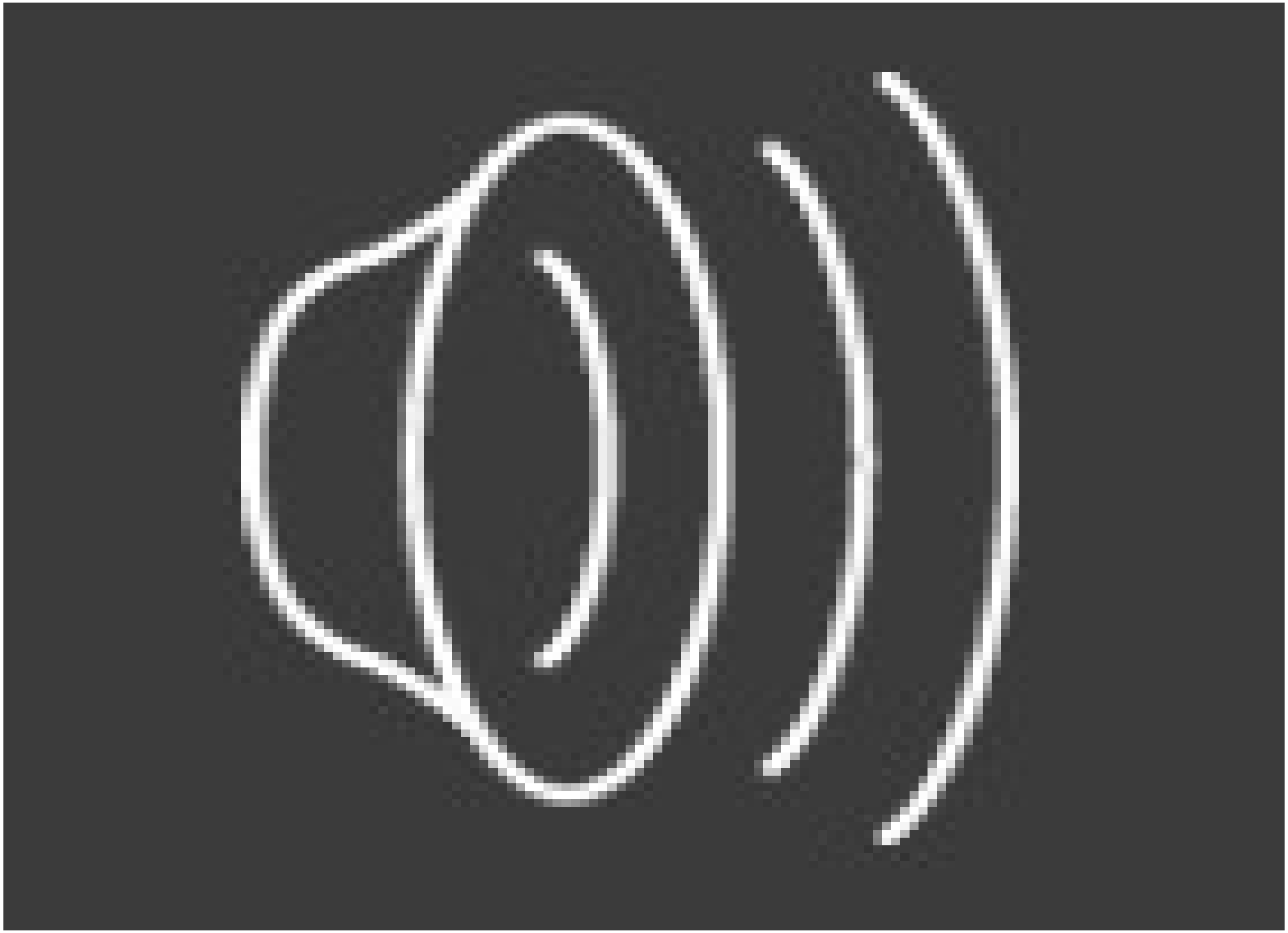
Recap: BFS/UCS



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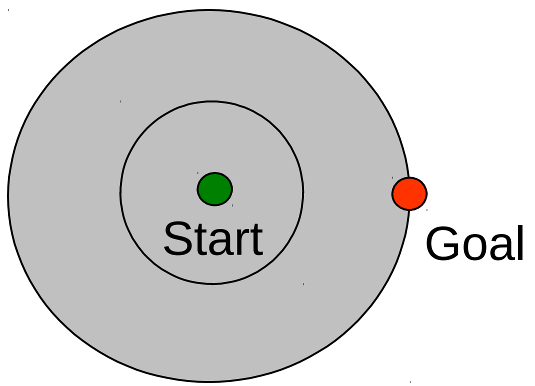
Recap: BFS/UCS

Notice that we search equally far in all directions...

It's like this



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Idea

Is it possible to use additional information to decide which direction to search in?

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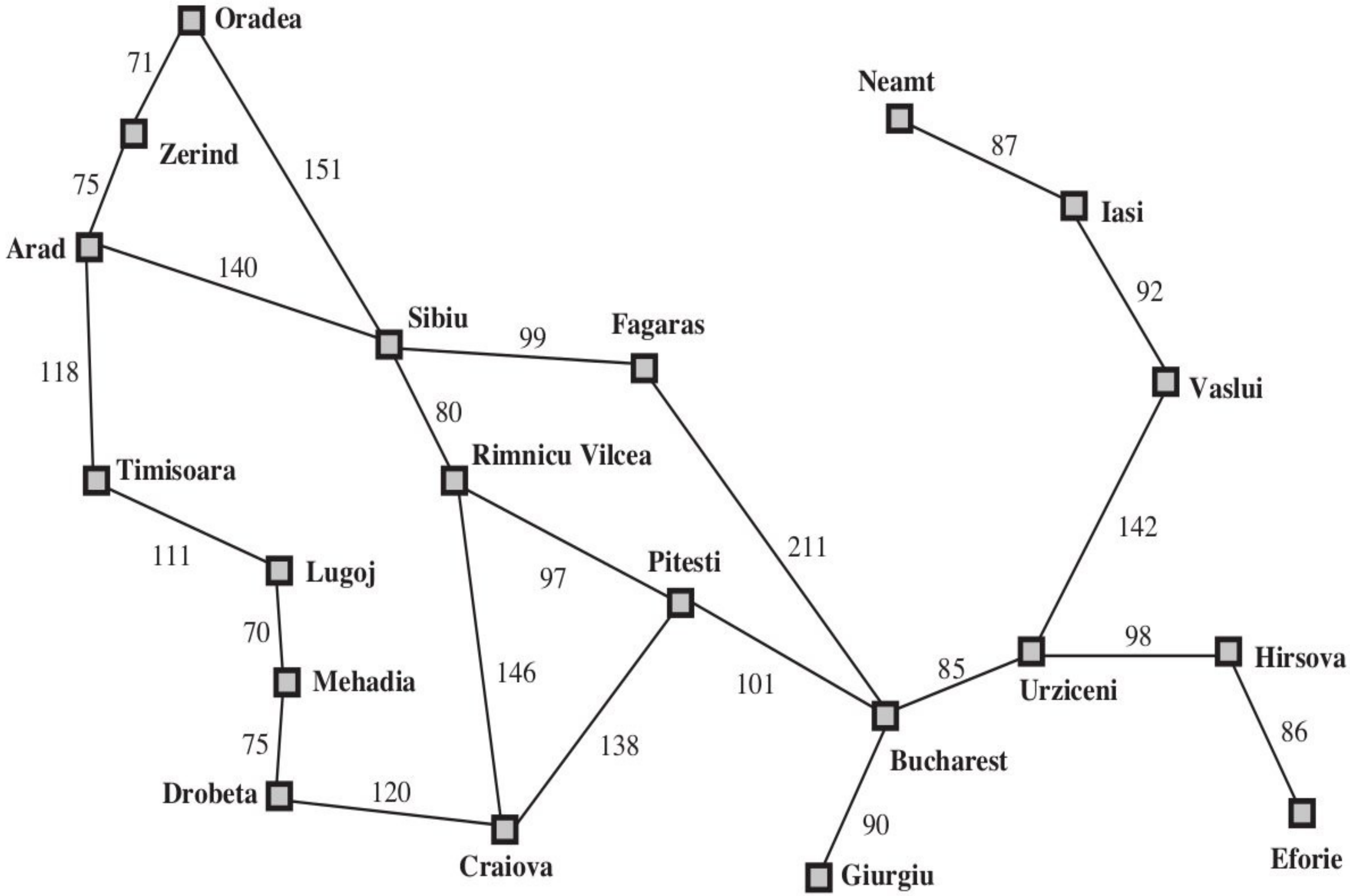
Yes!

Instead of searching in all directions, let's bias search in the direction of the goal.

Example



Arad	366
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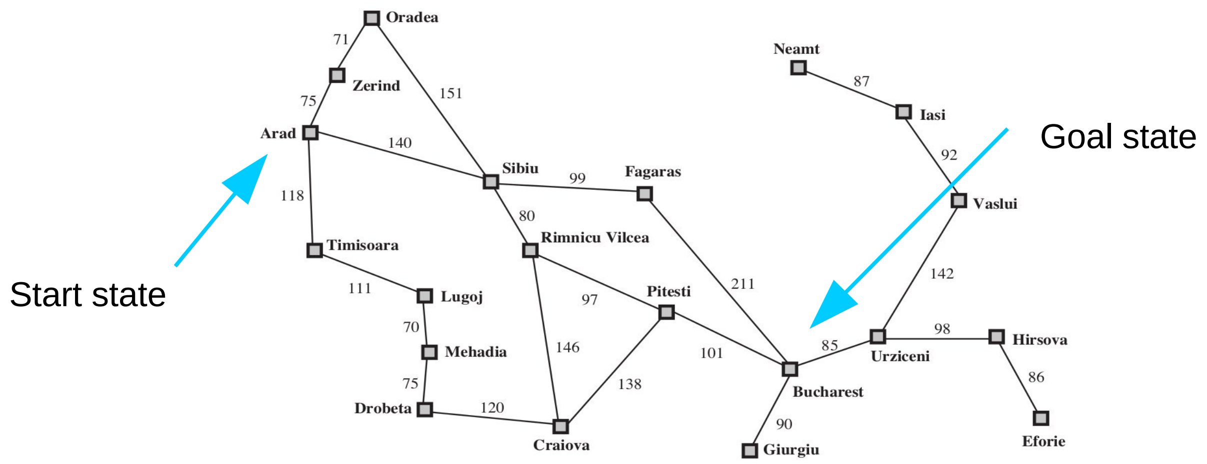
Stright-line distances to Bucharest

Example

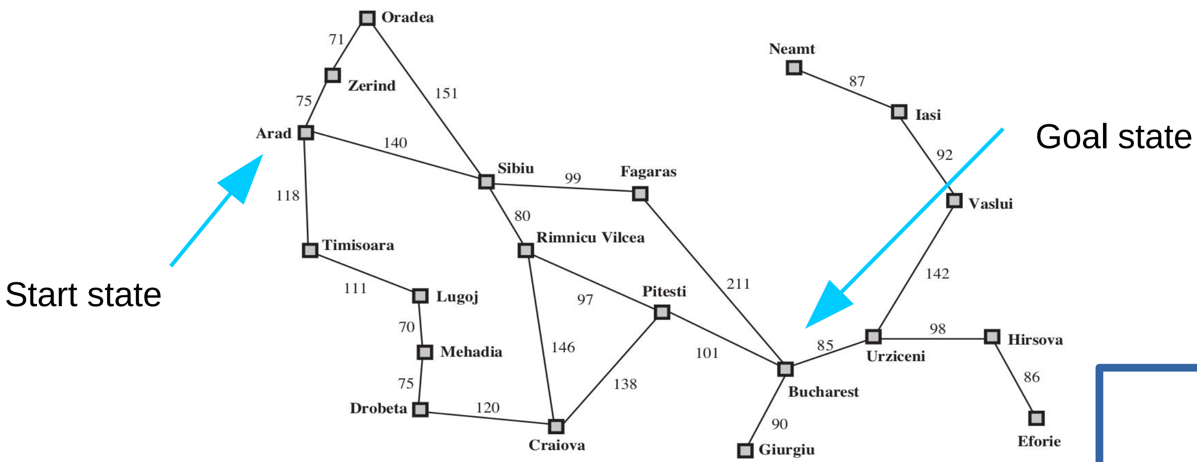


Expand states in order of their distance to the goal

– for each state that you put on the fringe: calculate straight-line distance to the goal

– expand the state on the fringe closest to the goal

Example



Heuristic: $h(s)$

Expand states in order of their distance to the goal

– for each state that you put on the fringe: calculate straight-line distance to the goal

– expand the state on the fringe closest to the goal

Greedy search

Greedy Search

Image: Adapted from Berkeley CS188 course notes (downloaded Summer 2015)

Greedy Search

Each time you expand a state, calculate the heuristic for each of the states that you add to the fringe.

– heuristic: $h(s)$ ⬅ i.e. distance to Bucharest

– on each step, choose to expand the state with the lowest heuristic value.

Greedy Search

This is like a guess about how far the state is from the goal

Each time you expand a state, calculate the heuristic for each of the states that you add to the fringe.

– heuristic: $h(s)$ ← i.e. distance to Bucharest

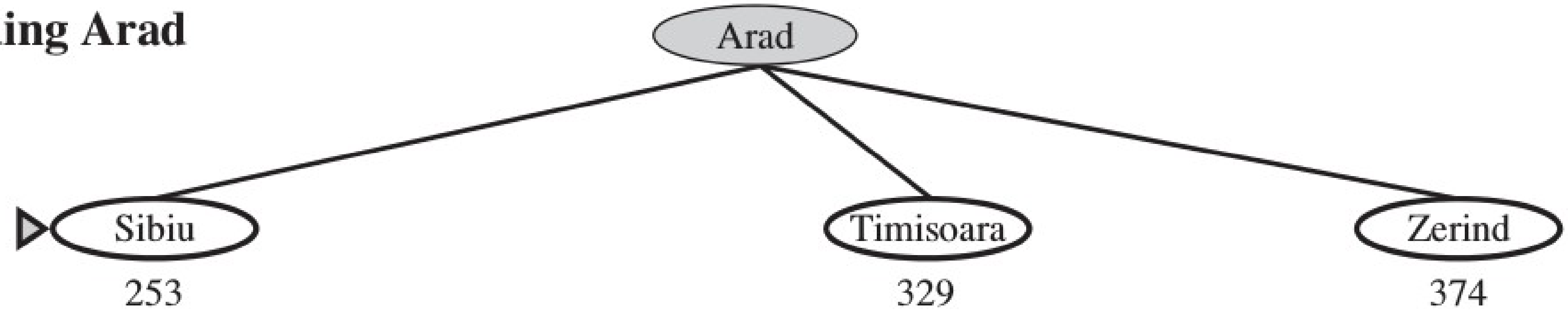
– on each step, choose to expand the state with the lowest heuristic value.

Example: Greedy Search

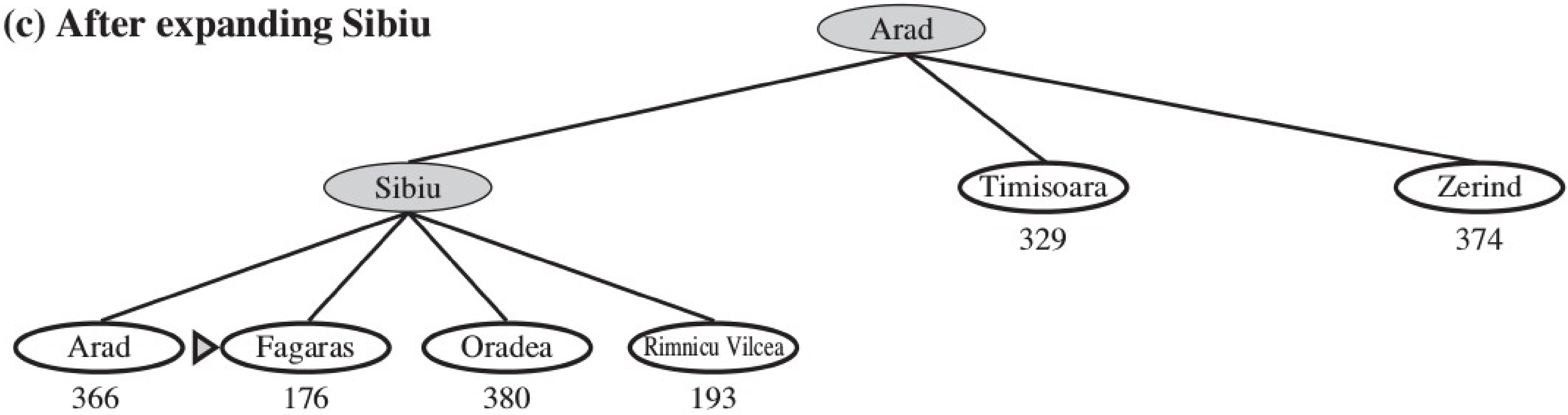
(a) The initial state

Example: Greedy Search

(b) After expanding Arad



Example: Greedy Search



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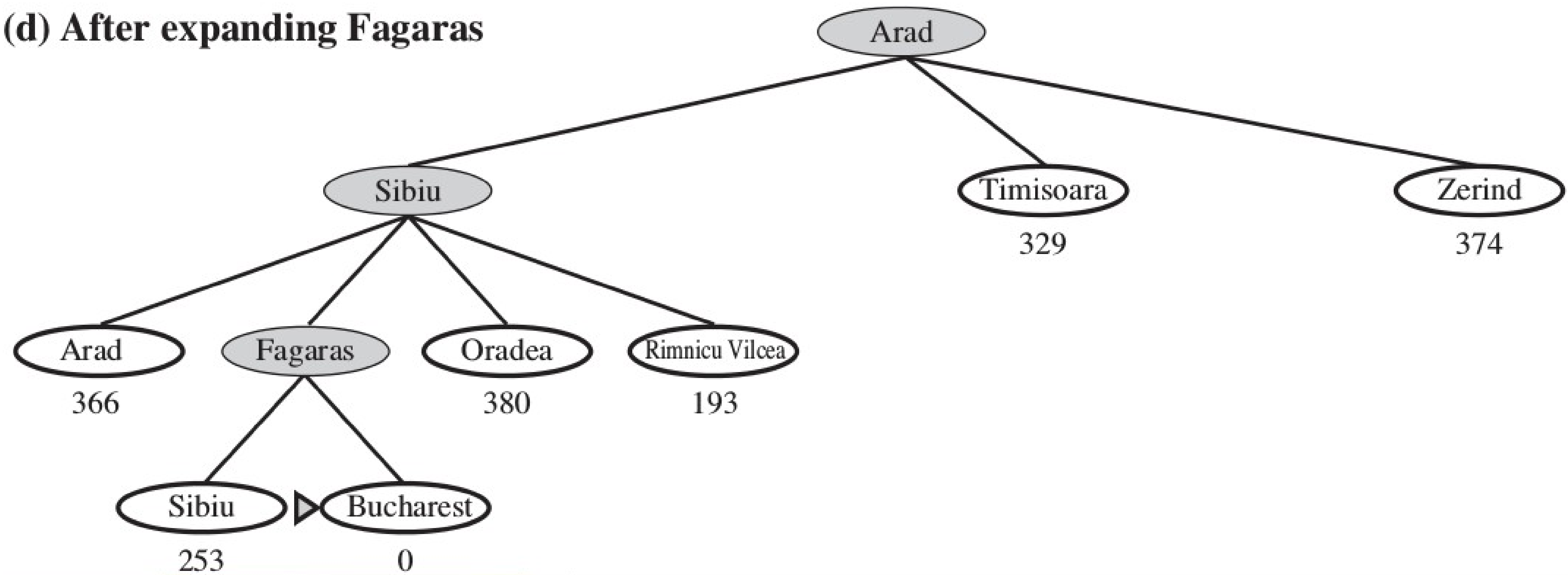
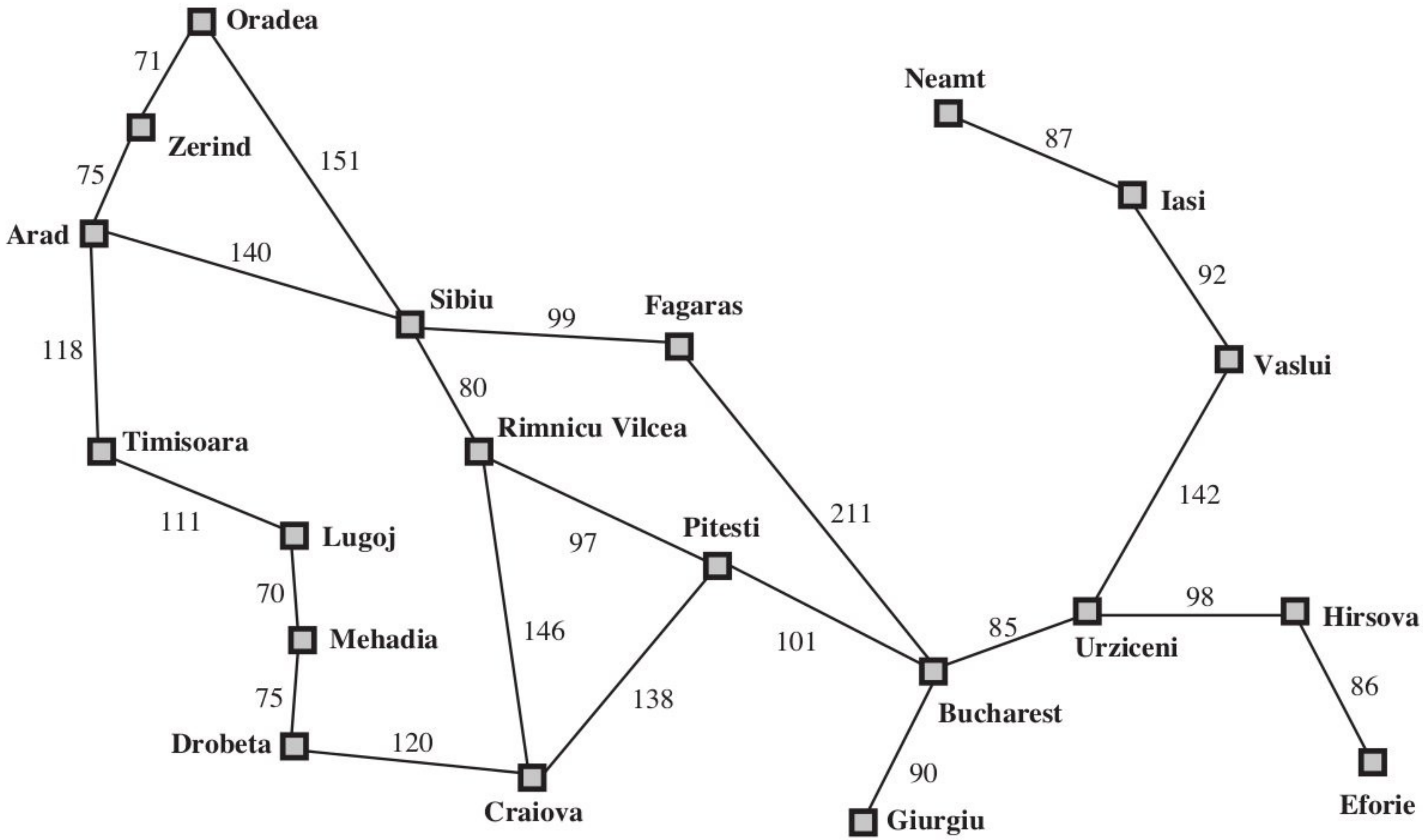


Path: A-S-F-B

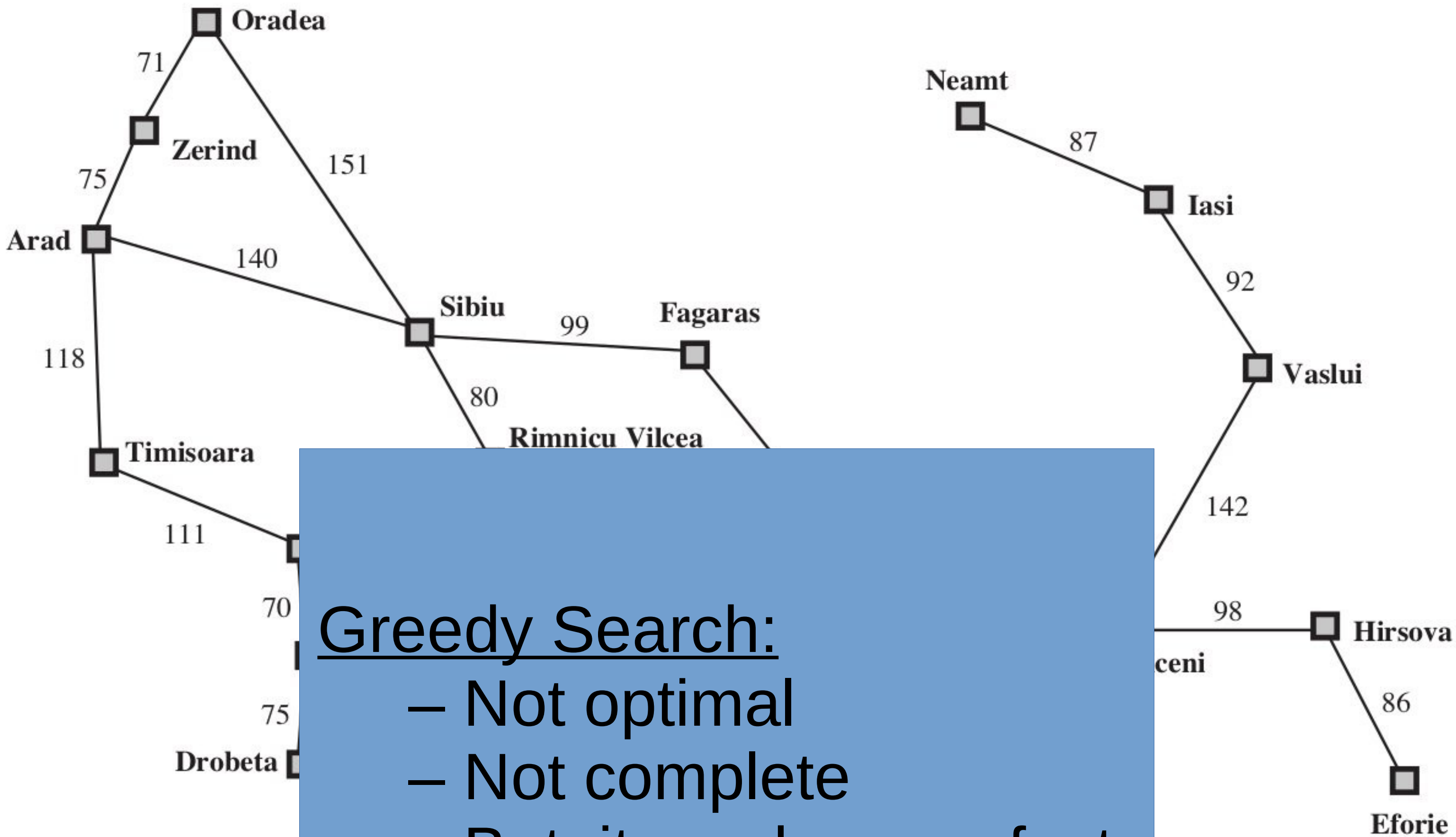
Example: Greedy Search



Path: A-S-F-B

<u>Notice that this is not the optimal path!</u>

Example: Greedy Search



<u>Greedy Search:</u>

– Not optimal
– Not complete
– But, it can be very fast

<u>Notice that this is not the optimal path!</u>

Greedy vs UCS

Greedy Search:

- Not optimal
- Not complete
- But, it can be very fast

UCS:

- Optimal
- Complete
- Usually very slow

Greedy vs UCS

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Can we combine greedy and UCS???

Greedy vs UCS

<u>Greedy Search:</u>

- Not optimal
- Not complete
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<u>UCS:</u>

- Optimal
- Complete
- Usually very slow

Can we combine greedy and UCS???

<u>YES: A*</u>

Greedy vs UCS

UCS

Image: Adapted from Berkeley CS188 course notes (downloaded Summer 2015)

Greedy vs UCS

UCS

Greedy

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Greedy vs UCS

UCS

Greedy

A*

Image: Adapted from Berkeley CS188 course notes (downloaded Summer 2015)

A*

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A*

s : a state

$g(s)$: minimum cost from start to

$h(s)$: heuristic at (*i.e.* an estimate of remaining cost-to-go)

UCS: expand states in order of $g(s)$

Greedy: expand states in order of $h(s)$

A*: expand states in order of $f(s) = g(s) + h(s)$

A*

<u>What is “cost-to-go”?</u>

s : a state

$g(s)$: minimum cost from start to s

$h(s)$: heuristic at s (*i.e.* an estimate of remaining cost-to-go)

<u>UCS</u>: expand states in order of $g(s)$

<u>Greedy</u>: expand states in order of $h(s)$

<u>A*</u>: expand states in order of $f(s) = g(s) + h(s)$

A*

s : a state

What is "cost-to-go"?
– minimum cost required to reach a goal state

$g(s)$: minimum cost from start to s

$h(s)$: heuristic at s (*i.e.* an estimate of remaining cost-to-go)

UCS: expand states in order of $g(s)$

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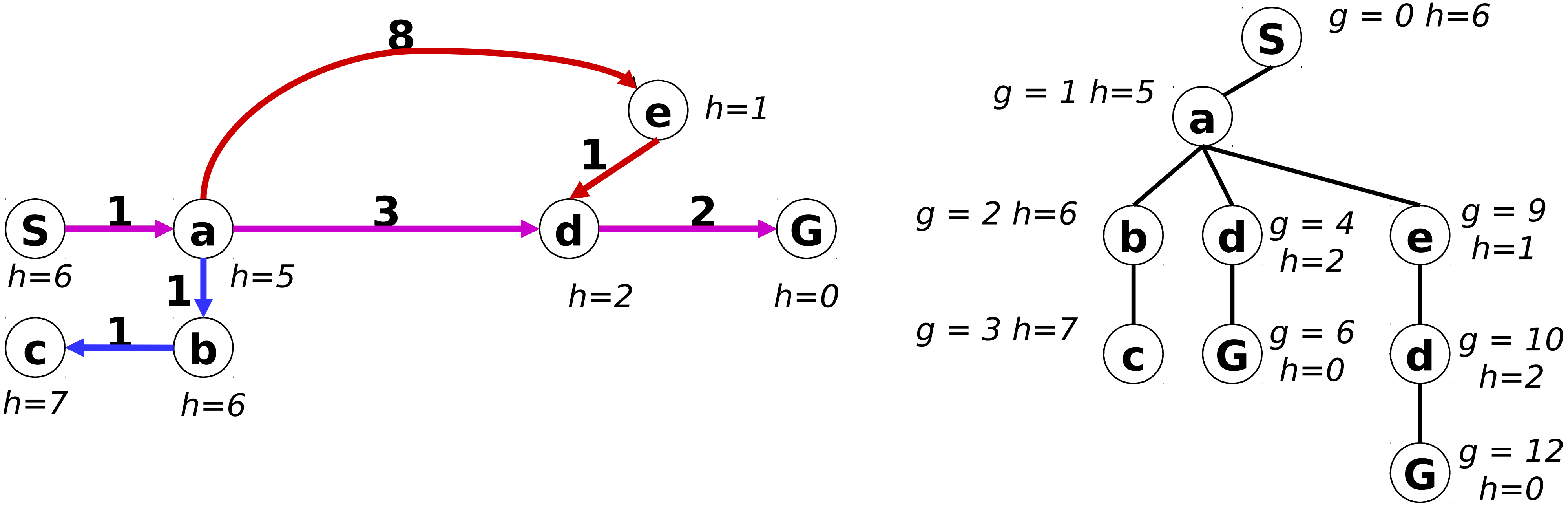

<u>A*</u>: expand states in order of $f(s) = g(s) + h(s)$

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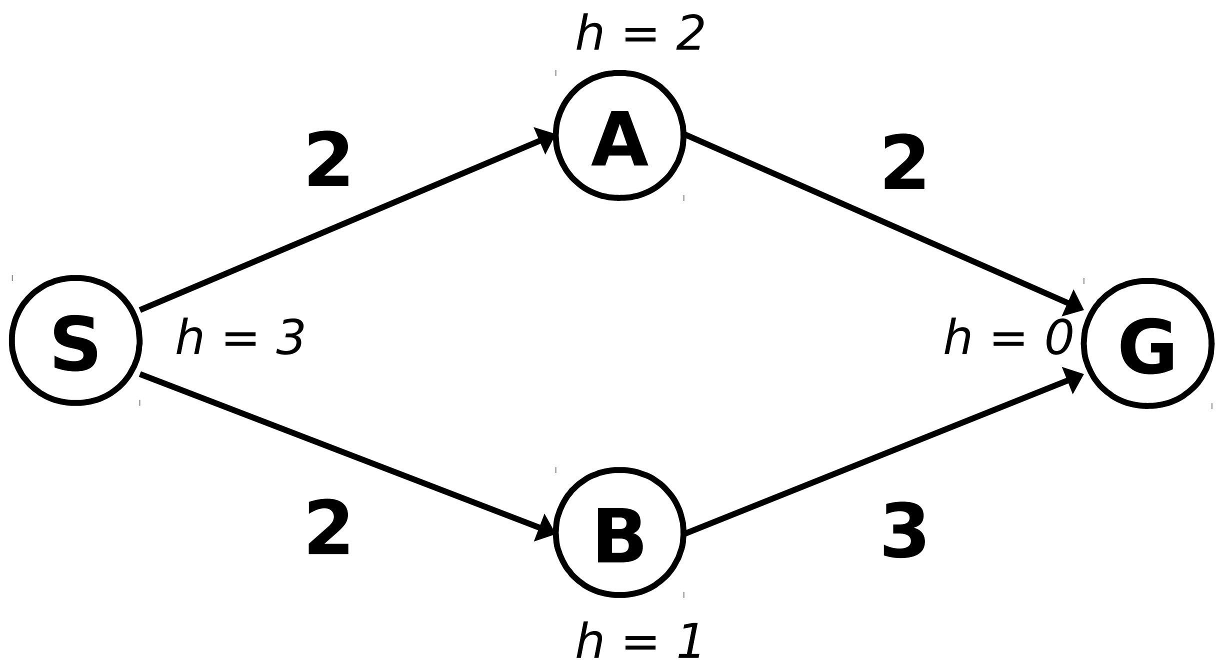
When should A* terminate?

Should we stop when we enqueue a goal?



No: only stop when we dequeue a goal

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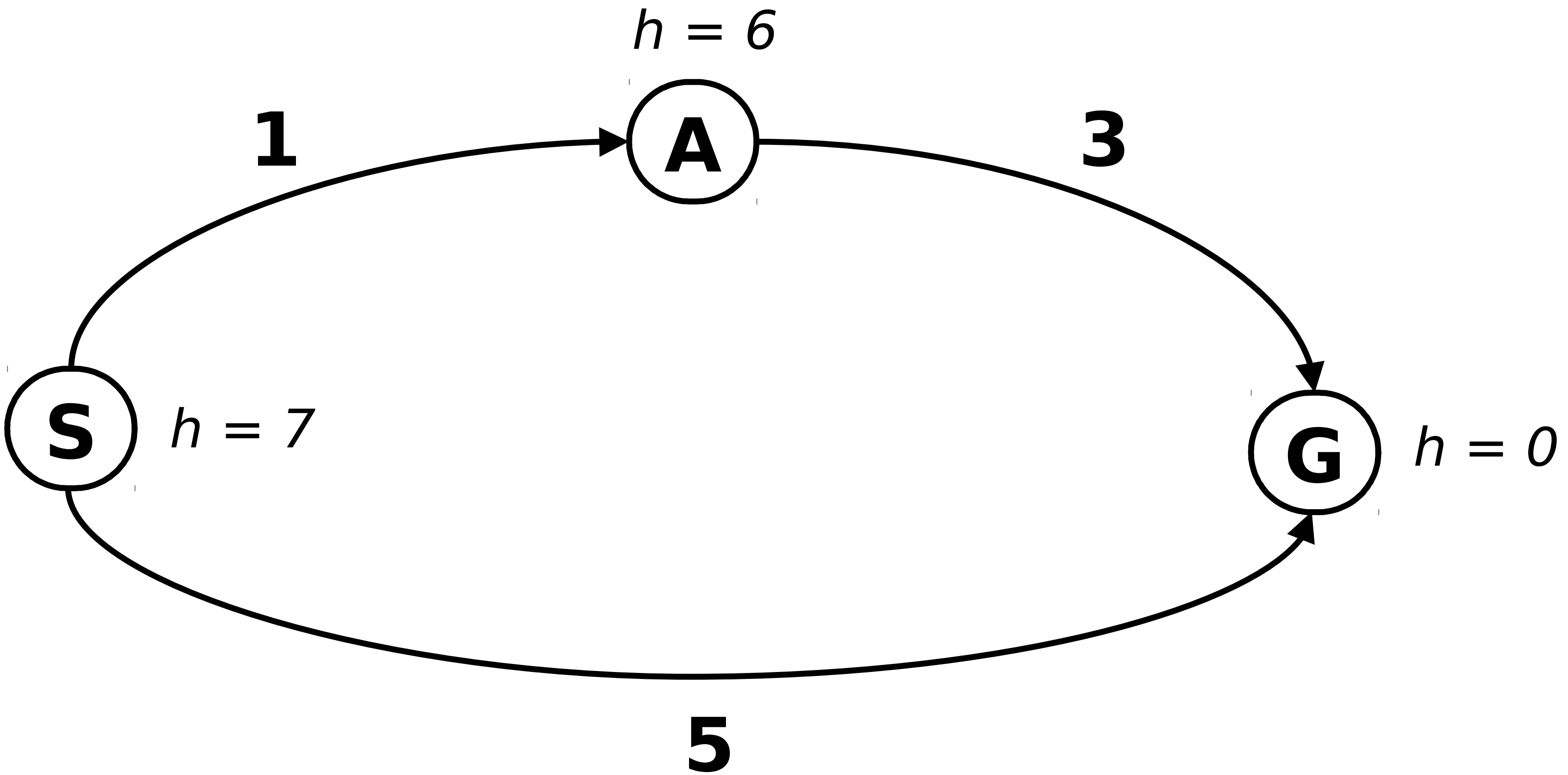
Is A* optimal?



What went wrong?
Actual cost-to-go < heuristic
The heuristic must be less than the actual cost-to-go!

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When is A* optimal?

It depends on whether we are using the tree search or the graph search version of the algorithm.

Recall:

– in tree search, we do not track the explored set
– in graph search, we do

Recall: Breadth first search (BFS)

```
function BREADTH-FIRST-SEARCH(problem) returns a solution, or failure
  node ← a node with STATE = problem.INITIAL-STATE, PATH-COST = 0
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  frontier ← a FIFO queue with node as the only element
  explored ← an empty set
  loop do
      if EMPTY?(frontier) then return failure
      node ← POP(frontier)  /* chooses the shallowest node in frontier */
      add node.STATE to explored
      for each action in problem.ACTIONS(node.STATE) do
          child ← CHILD-NODE(problem, node, action)
          if child.STATE is not in explored or frontier then
              if problem.GOAL-TEST(child.STATE) then return SOLUTION(child)
              frontier ← INSERT(child, frontier)
```

Figure 3.11 Breadth-first search on a graph.

What is the purpose of the *explored* set?

When is A* optimal?

It depends on whether we are using the <u>tree search</u> or the <u>graph search</u> version of the algorithm.

Optimal if h is <u>consistent</u>

Optimal if h is <u>admissible</u>

When is A* optimal?

It depends on whether we are using the tree search or the graph search version of the algorithm.

Optimal if h is consistent

– h(s) is an underestimate of the cost of each arc.

Optimal if h is admissible

– *h(s)* is an underestimate of the true cost-to-go.

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<u>What is "cost-to-go"?</u>

– minimum cost required to reach a goal state

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Optimal if h is consistent

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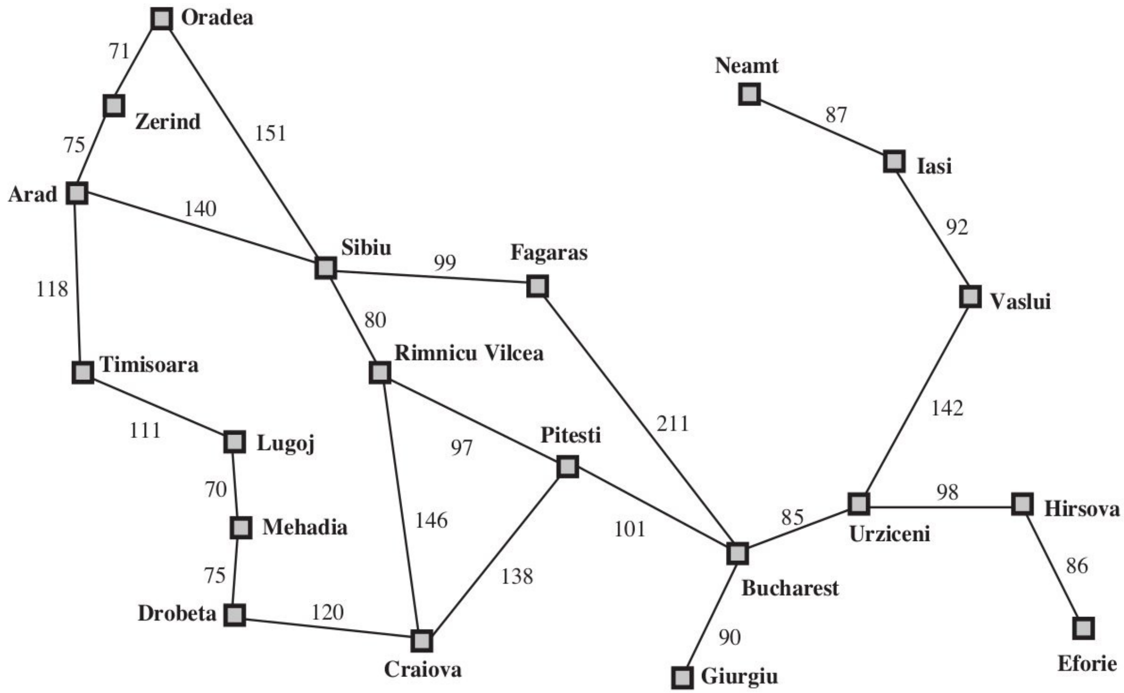
Optimal if h is admissible

– *h(s)* is an underestimate of the true cost-to-go.

More on this later...

Admissibility: Example



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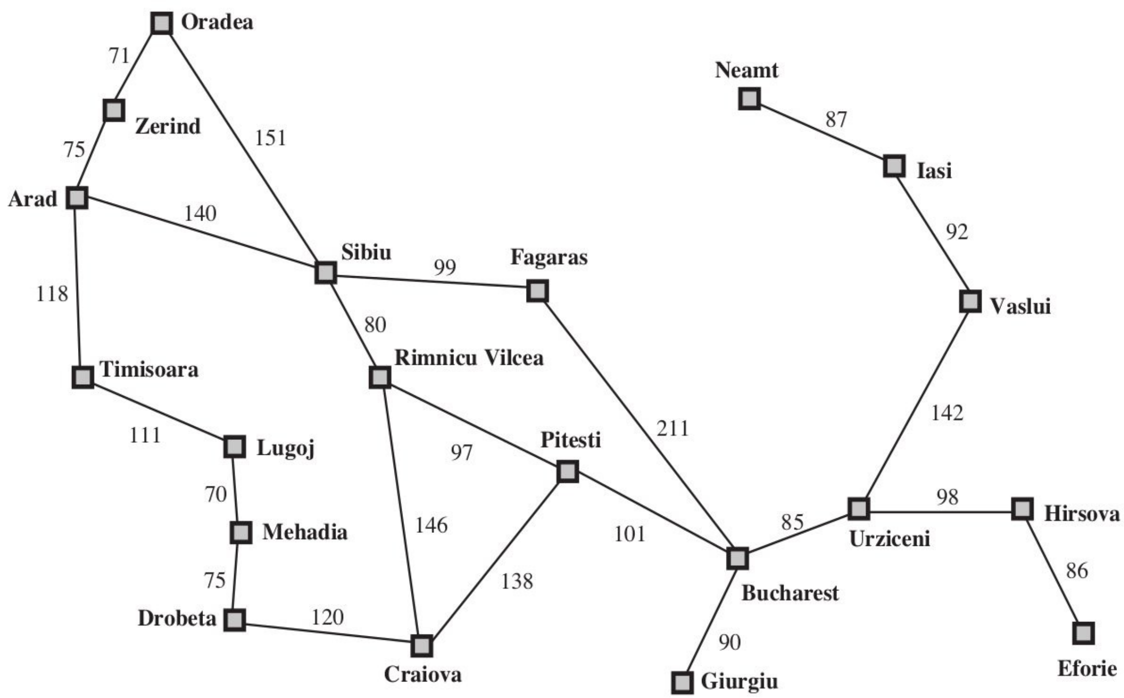
Stright-line distances to Bucharest

h(s) = straight-line distance to goal state (Bucharest)

Admissibility



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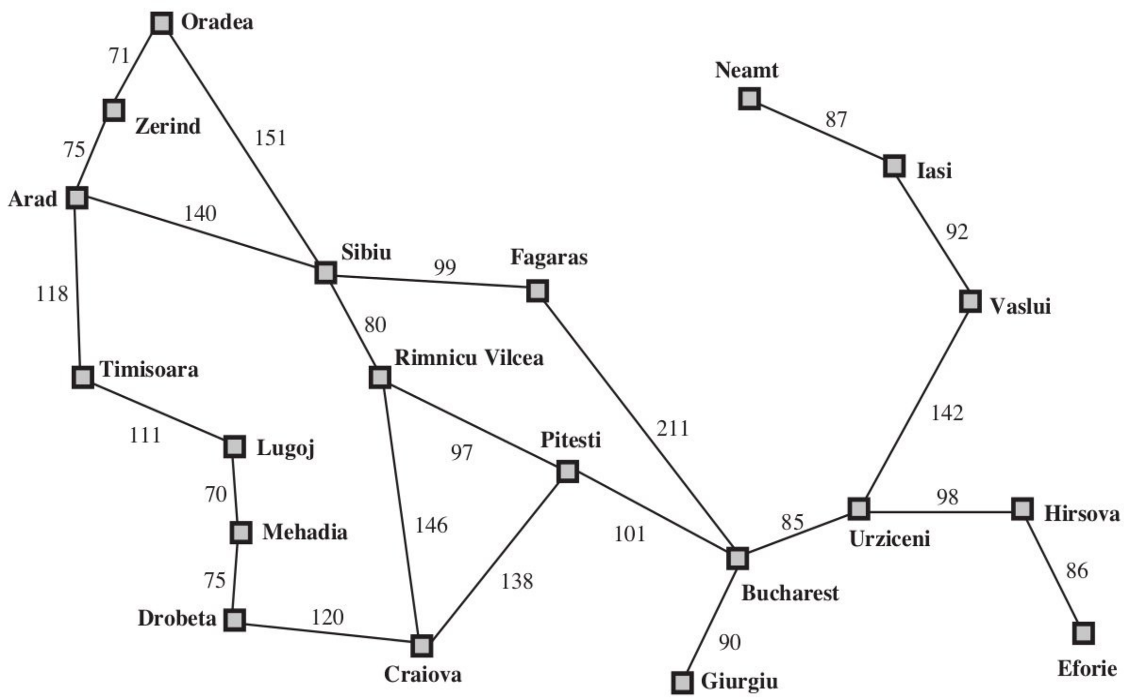
Is this heuristic admissible???

Admissibility



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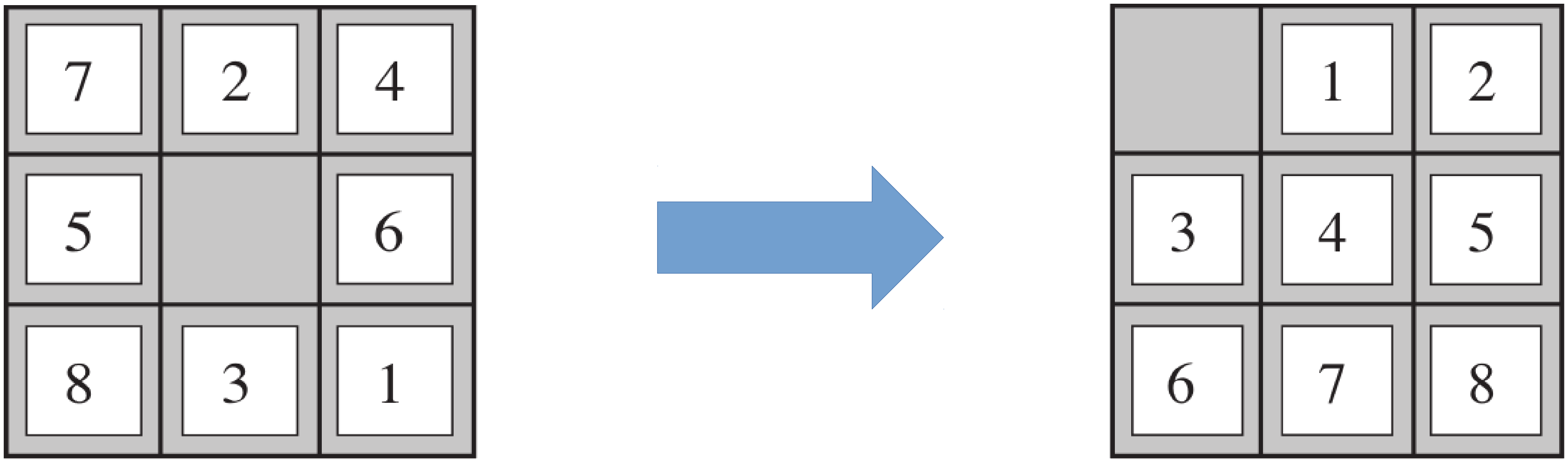
Is this heuristic admissible???
YES! Why?

Admissibility: Example



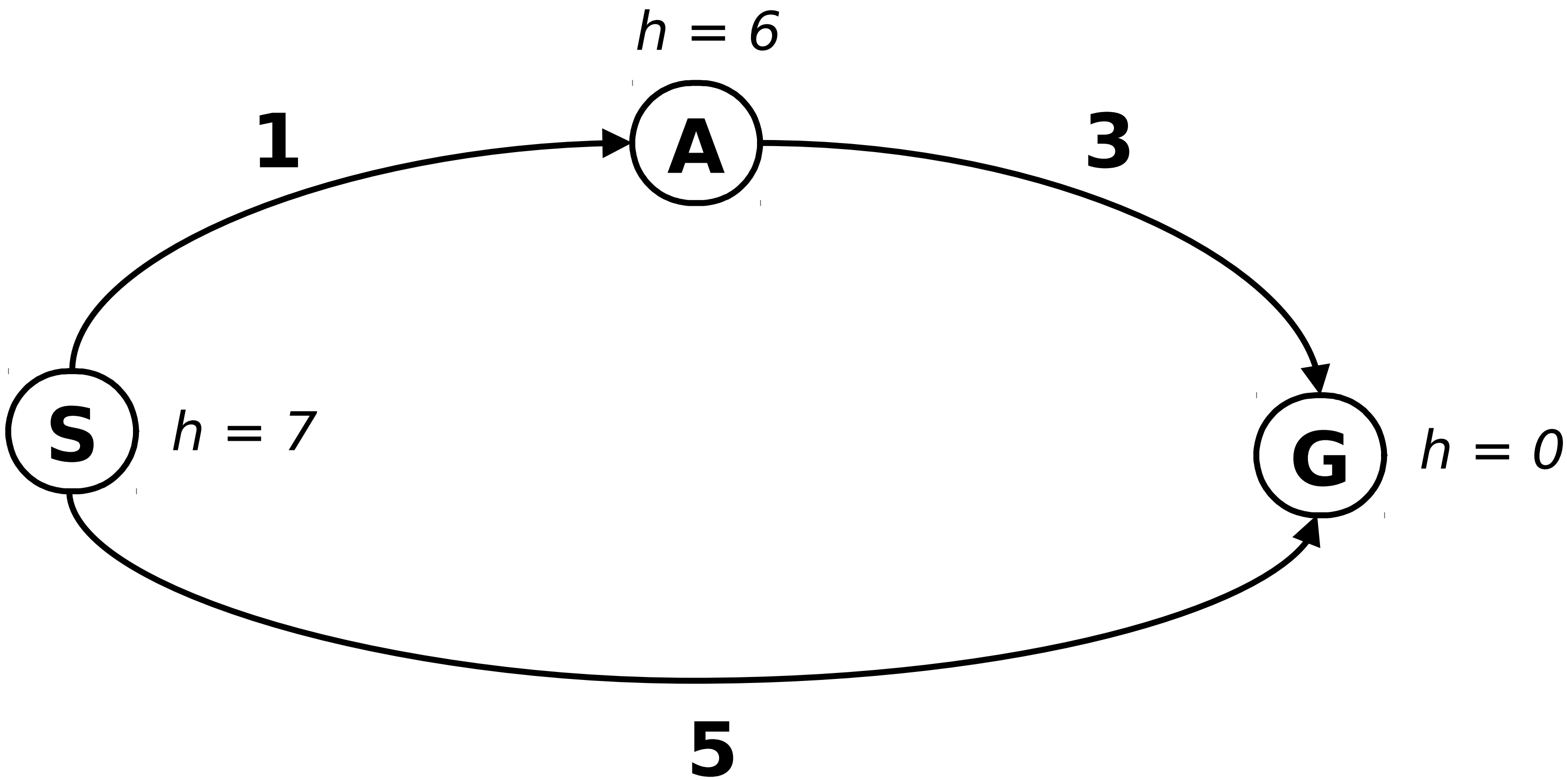
Start state

Goal state

h(s) = ?

Can you think of an admissible heuristic for this problem?

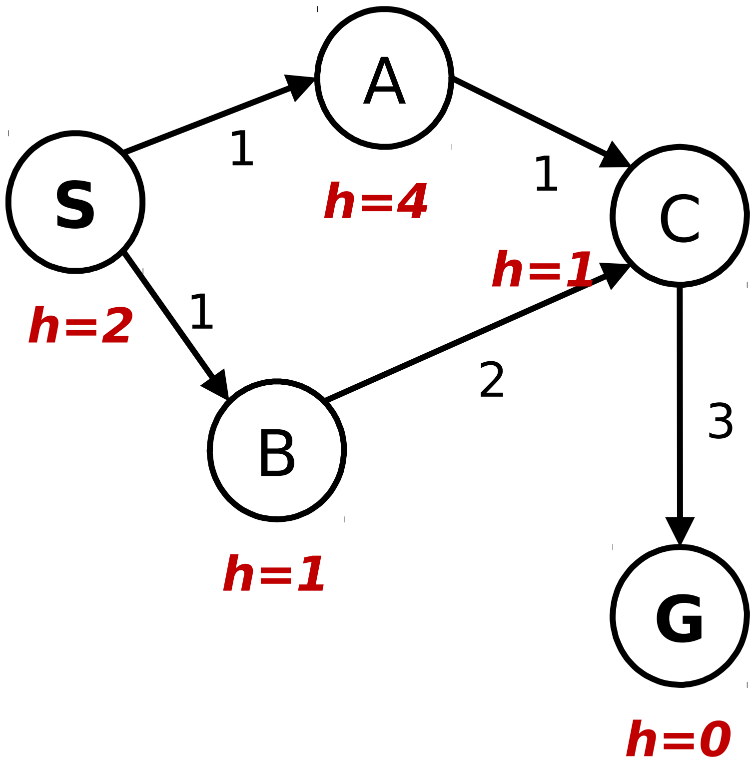
Admissibility



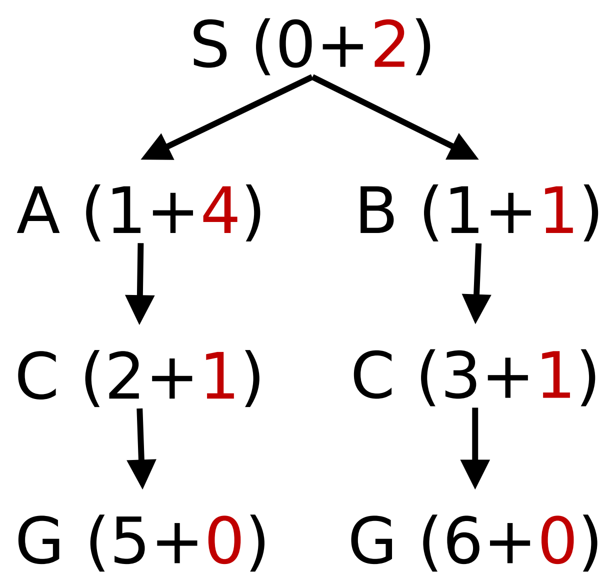
Why isn't this heuristic admissible?

Consistency

State space graph



Search tree



What went wrong?

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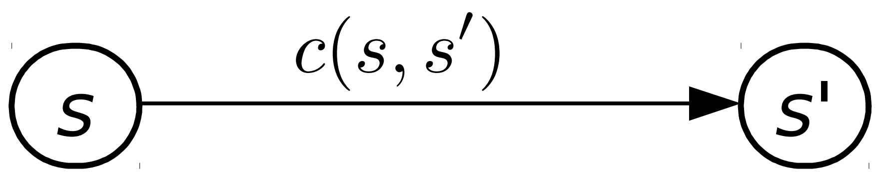
Consistency

$$h(s) \leq c(s, s') + h(s')$$

Cost of going from *s* to *s′*



Consistency

$$h(s) \leq c(s, s') + h(s')$$

$$h(s) - h(s') \leq c(s, s')$$ ⟵ Rearrange terms

Consistency

$$h(s) \leq c(s, s') + h(s')$$

$$\underbrace{h(s) - h(s')}_{} \leq c(s, s')$$

Cost of going from s to s'
implied by heuristic

Actual cost of
going from *s* to *s'*

Consistency

$$h(s) \leq c(s, s') + h(s')$$

$$h(s) - h(s') \leq c(s, s')$$

Cost of going from s to s'
implied by heuristic

Actual cost of
going from *s* to *s'*

Consistency

$$f(s) = g(s) + h(s)$$

Consistency implies that the “f-cost” never decreases along any path to a goal state.
– the optimal path gives a goal state its lowest f-cost.

A* expands states in order of their f-cost.

Given any goal state, A* expands states that reach the goal state optimally before expanding states the reach the goal state suboptimally.

Consistency implies admissibility

Suppose: $\forall s_t, s_{t+1} : h(s_t) \leq c(s_t, s_{t+1}) + h(s_{t+1})$

Then: $h(s_{T-1}) \leq c(s_{T-1}, s_T) + h(s_T)$

Consistency implies admissibility

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admissible

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admissible admissible

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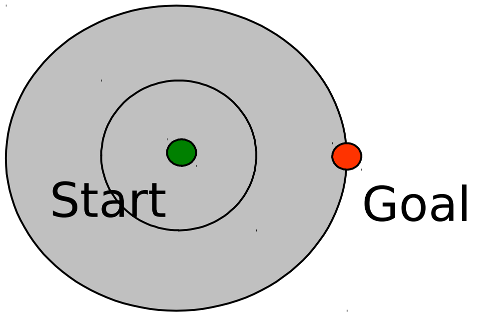
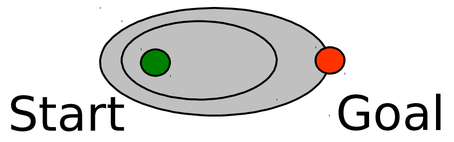
• • •

A* vs UCS

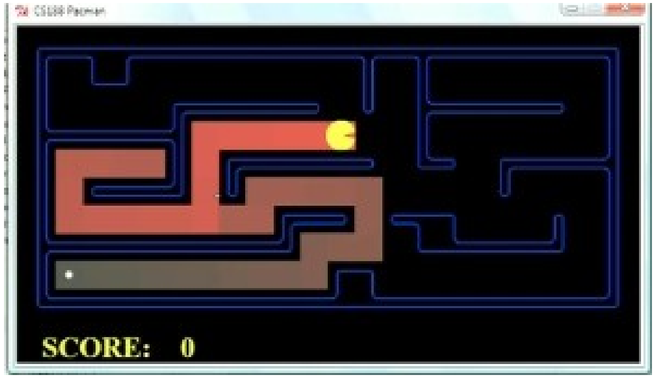
- Uniform-cost expands equally in all “directions”



- A* expands mainly toward the goal, but does hedge its bets to ensure optimality



A* vs UCS



Greedy

UCS

A*

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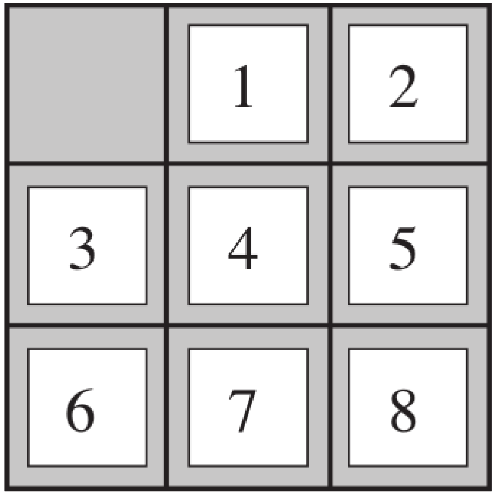
Choosing a heuristic

The right heuristic is often problem-specific.

But it is very important to select a good heuristic!

Choosing a heuristic



Consider the 8-puzzle:

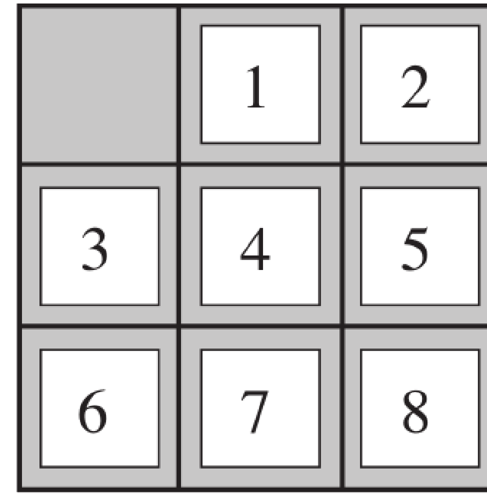
h_1 : number of misplaced tiles

h_2 : sum of manhattan distances between each tile and its goal.

How much better is h_2 ?

Choosing a heuristic

<u>Consider the 8-puzzle:</u>



h_1 : number of misplaced tiles

h_2 : sum of manhattan distances between each tile and its goal.

Average # states expanded on a random depth-24 puzzle:

$$A^*(h_1) = 39k$$

$$A^*(h_2) = 1.6k$$

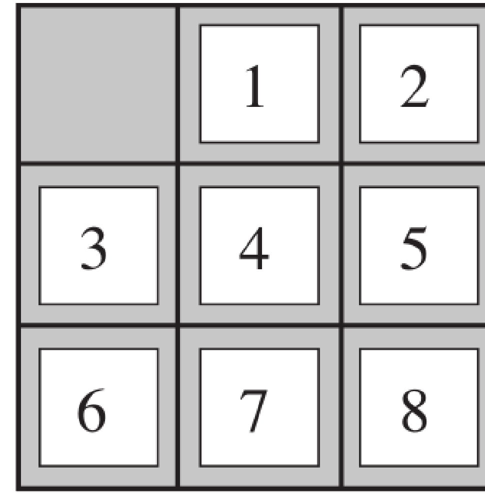
$$IDS = 3.6M$$ (by depth 12)

Choosing a heuristic

<u>Consider the 8-puzzle:</u>

h_1 : number of misplaced tiles

h_2 : sum of manhattan distances between each tile and its goal.

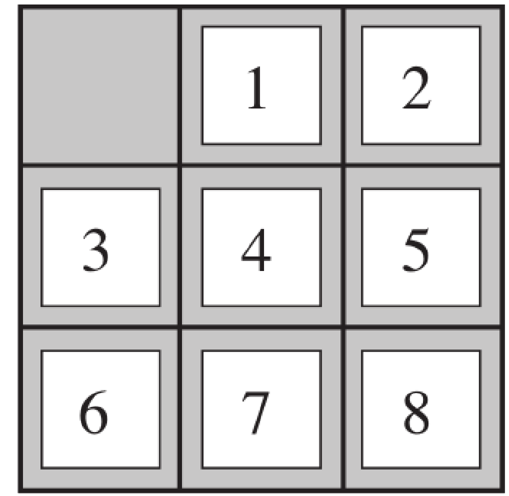


zle:

So, getting the heuristic right can speed things up by multiple orders of magnitude!

$IDS = 3.6M$ (by depth 12)

Choosing a heuristic



Consider the 8-puzzle:

h_1 : number of misplaced tiles

h_2 : sum of manhattan distances between each tile and its goal.

Why not use the actual cost to goal as a heuristic?

How to choose a heuristic?

Nobody has an answer that always works.

A couple of best-practices:
– solve a relaxed version of the problem
– solve a subproblem