Constraint Satisfaction Problems

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Some images and slides are used from:
1. CS188 UC Berkeley
2. RN, AIMA

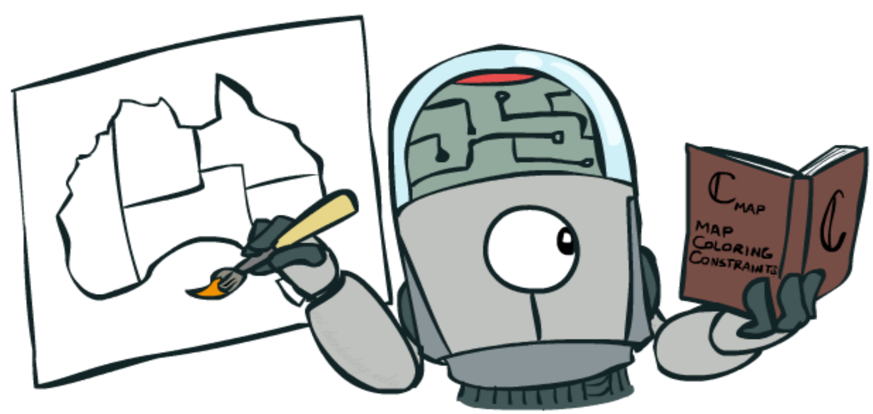
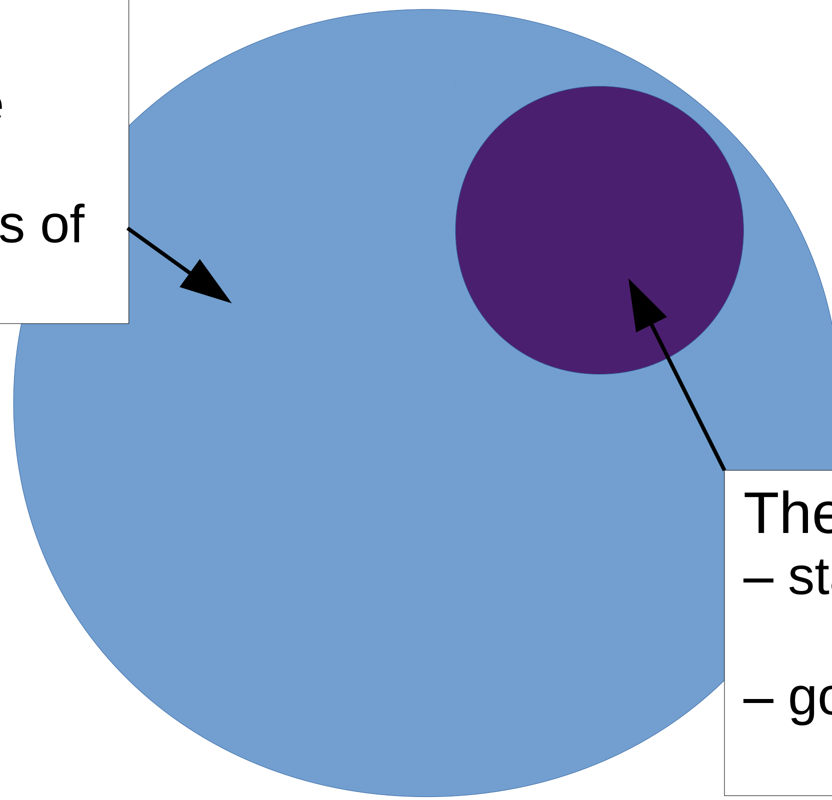


Image: Berkeley CS188 course notes (downloaded Summer 2015)

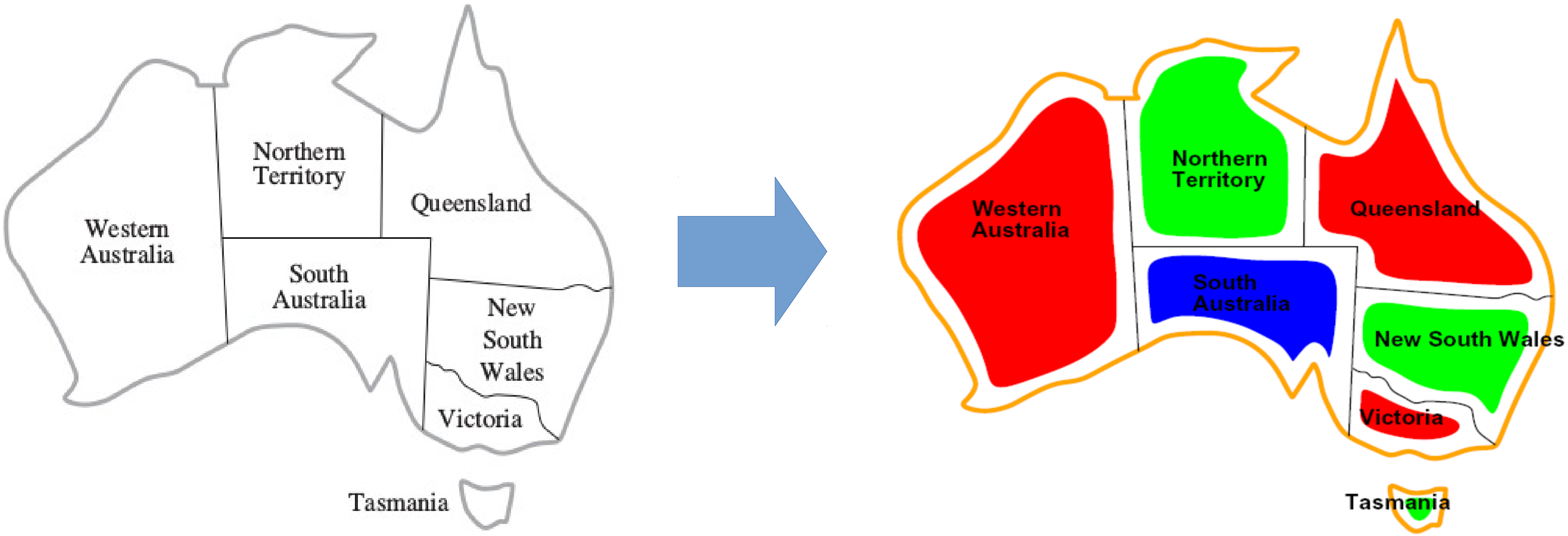
What is a CSP?

CSPs $\subseteq$ All search problems



<u>A CSP is defined by:</u>
1. a set of variables and their associated domains
2. a set of constraints that must be satisfied.

CSP example: map coloring



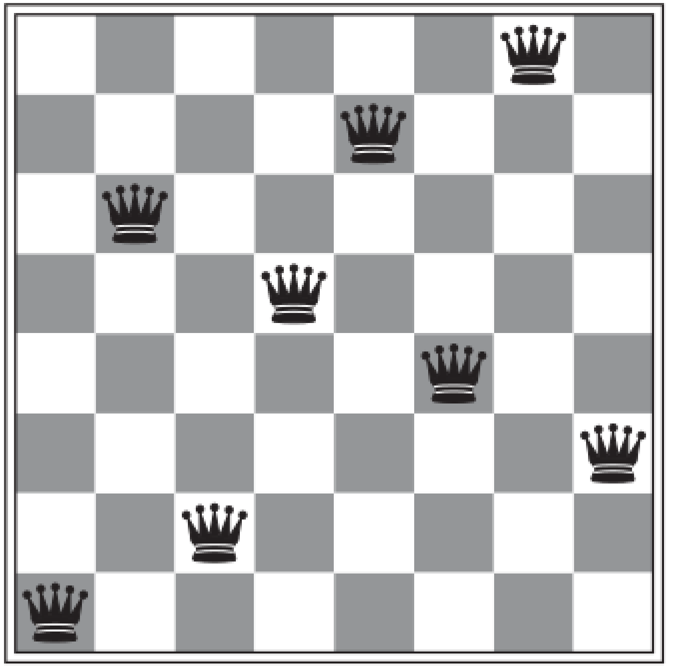
Problem: assign each territory a color such that no two adjacent territories have the same color

Variables: $X = \{WA, NT, Q, NSW, V, SA, T\}$

Domain of variables: $D = \{r, g, b\}$

Constraints: $C = \{SA \neq WA, SA \neq NT, SA \neq Q, \ldots\}$

CSP example: n-queens



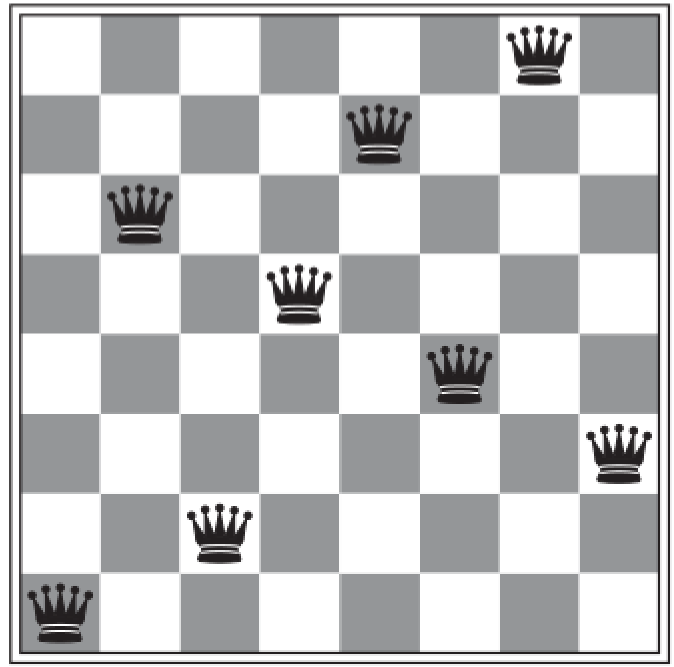
Problem: place n queens on an nxn chessboard such that no two queens threaten each other

Variables: $X = ?$

Domain of variables: $D = ?$

Constraints: $C = ?$

CSP example: n-queens



<u>Problem:</u> place n queens on an nxn chessboard such that no two queens threaten each other

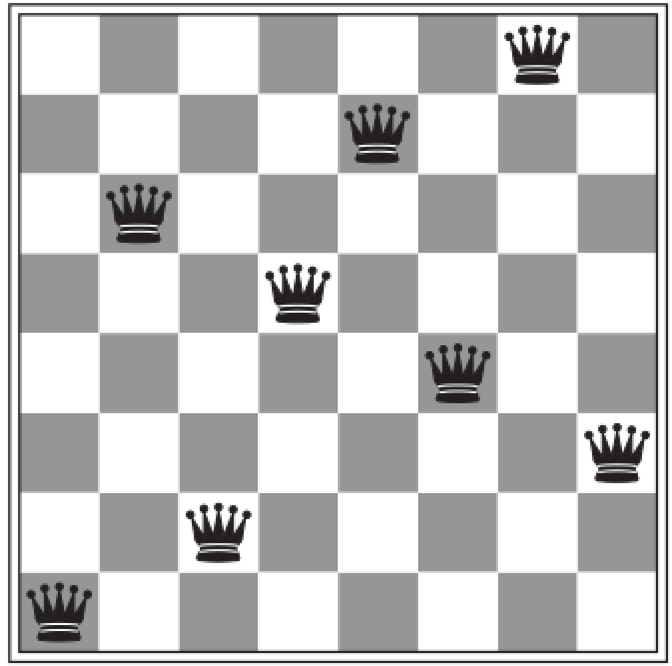
Variables: $X =$ One variable for every square

Domain of variables: $D =$ Binary

Constraints: $C =$ Enumeration of each possible disallowed configuration

– why is this a bad way to encode the problem?

CSP example: n-queens



<u>Problem:</u> place n queens on an nxn chessboard such that no two queens th

Variables

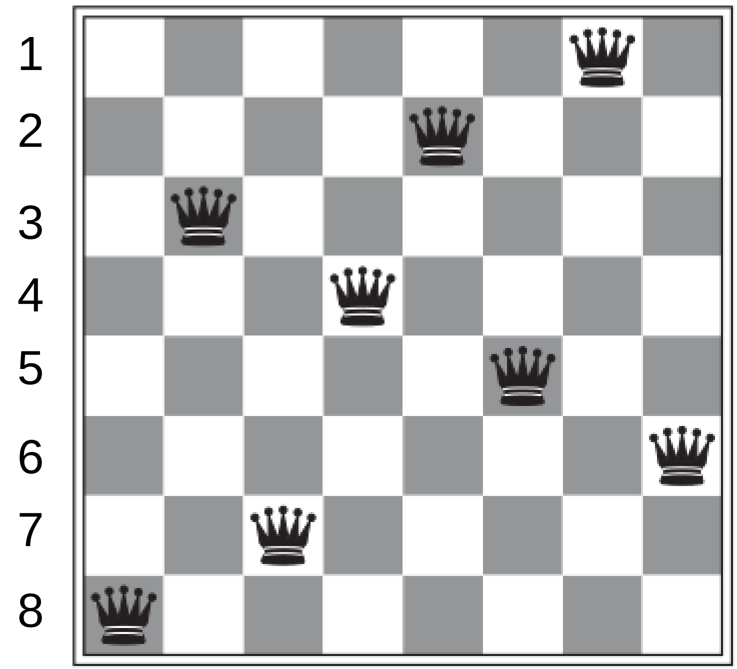
Is there a better way?

Domain c

Constrain ved configuration

– why is this a bad way to encode the problem?

CSP example: n-queens



<u>Problem:</u> place n queens on an nxn chessboard such that no two queens threaten each other

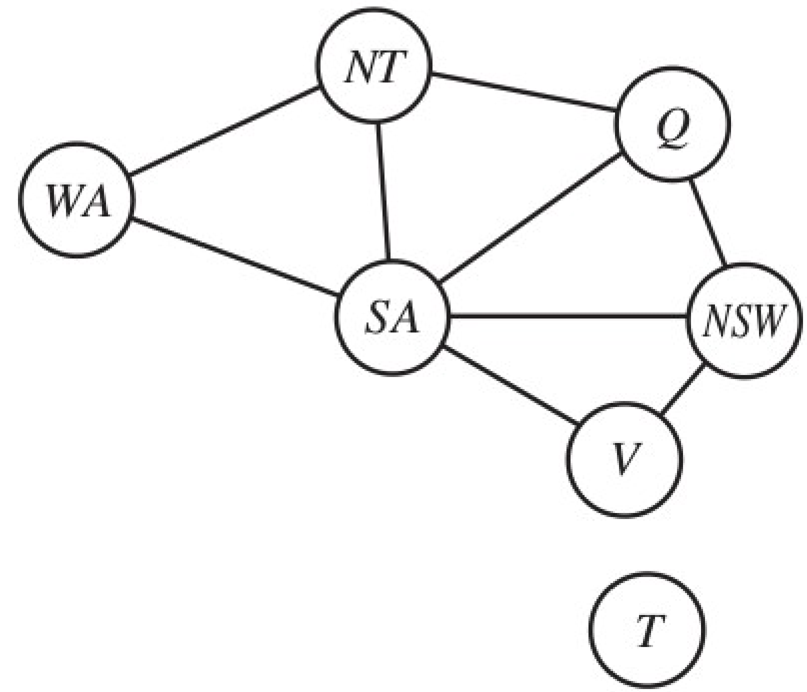
Variables: $X =$ One variable for each row

Domain of variables: $D =$ A number between 1 and 8

Constraints: $C =$ Enumeration of disallowed configurations

– why is this representation better?

The constraint graph



Variables represented as nodes (i.e. as circles)

Constraint relations represented as edges

– map coloring is a binary CSP, so it's easier to represent...

A harder CSP to represent: Cryptarithmetic

- Variables:

  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

- Domains:

  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

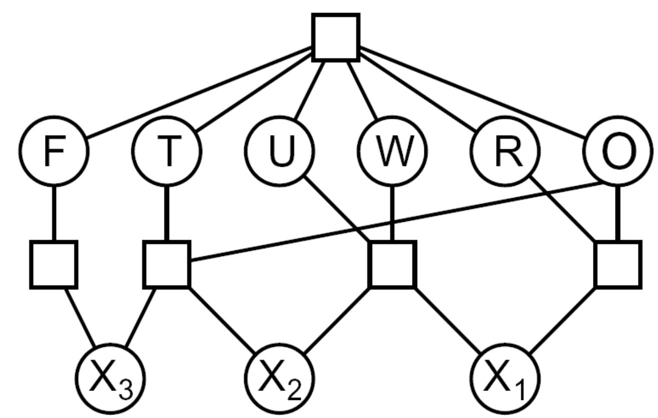
- Constraints:

  $\mathrm{alldiff}(F, T, U, W, R, O)$

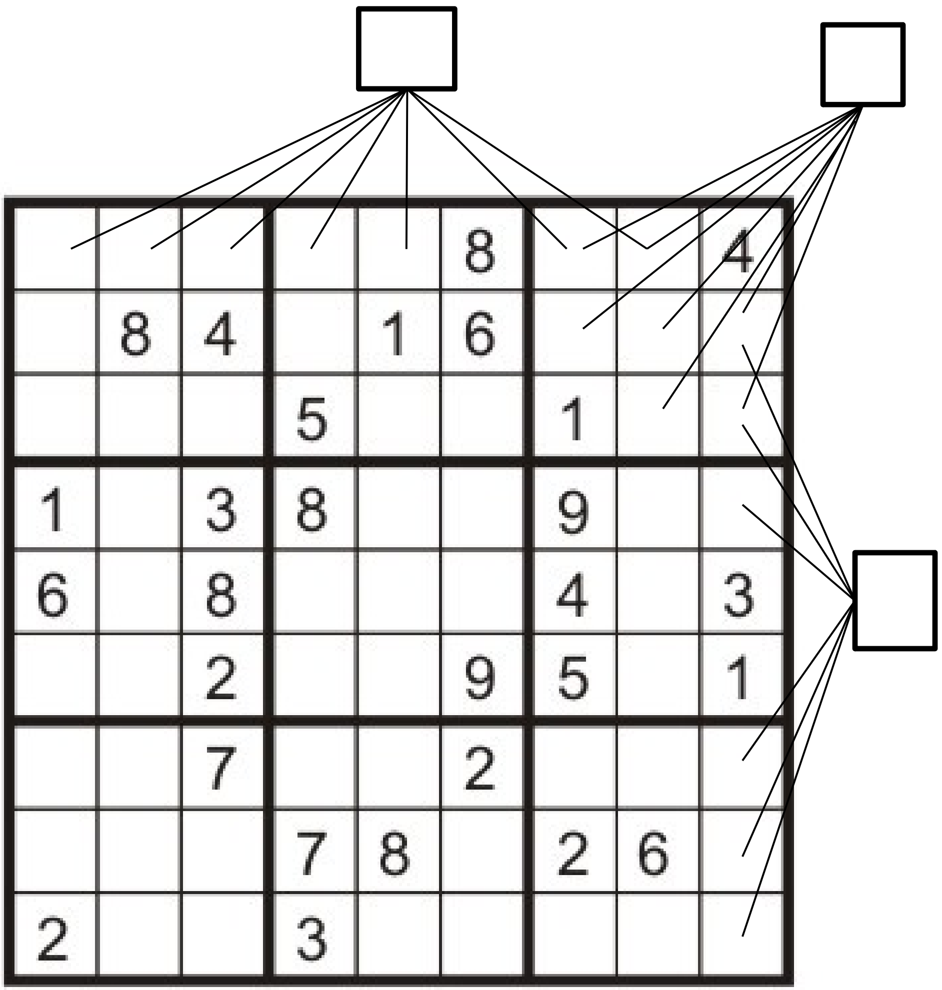
  $O + O = R + 10 \cdot X_1$

  $\cdots$

```
    T W O
+   T W O
---------
  F O U R
```



Another example: sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

9-way alldiff for each column

9-way alldiff for each row

9-way alldiff for each region

(or can have a bunch of pairwise inequality constraints)

Naive solution: apply BFS, DFS, A*, ...

Which would be better: BFS, DFS, A*?

– remember: it doesn't know if it reached
a goal until all variables are assigned ...

Naive solution: apply BFS, DFS, A*, ...

_ _ _ _ _ _ _

R _ _ _ _ _ _

R G _ _ _ _ _

R G R _ _ _ _

⋮ ⋮

R G R R R R R

How many leaf nodes are expanded in the worst case?

Naive solution: apply BFS, DFS, A*, ...

_ _ _ _ _ _ _

R _ _ _ _ _ _

R G _ _ _ _ _

R G R _ _ _ _

⋮

⋮

R G R R R R R

How many leaf nodes are expanded in the worst case? $3^7 = 2187$

Naive solution: apply BFS, DFS, A*, ...

_ _ _ _ _ _ _

R _ _ _ _ _ _

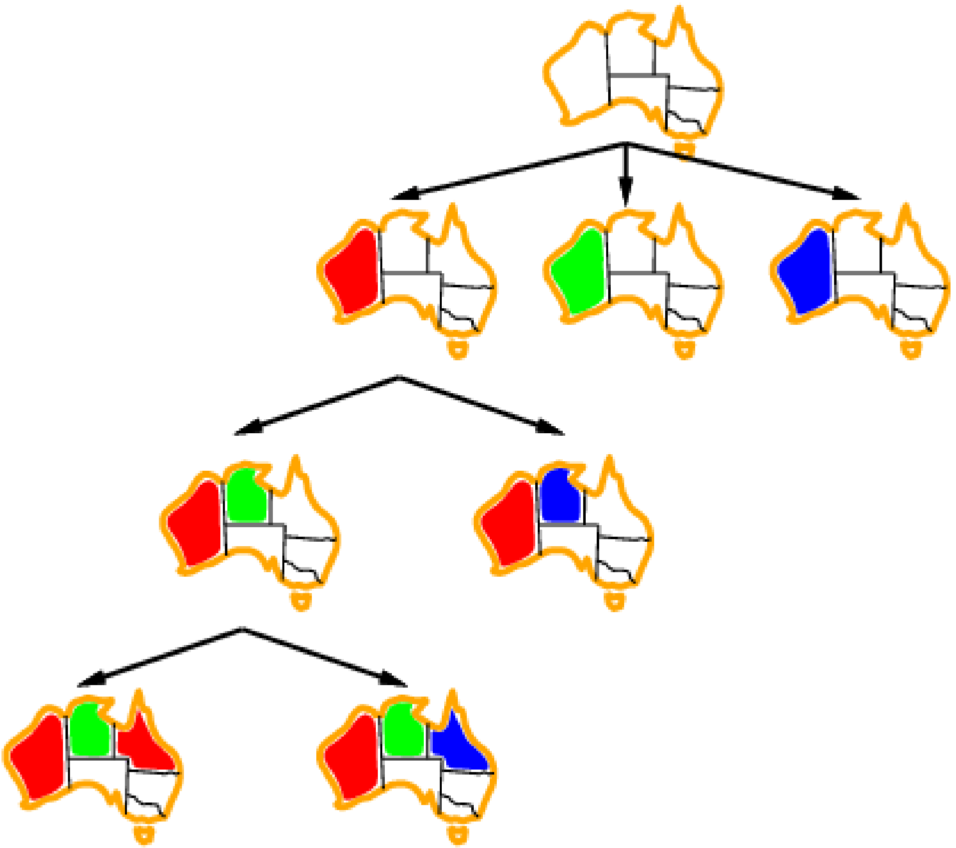
This sucks.
How can we improve it?

R G R R R R R

How many leaf nodes are expanded in the worst case? $3^7 = 2187$

Backtracking search

When a node is expanded, check that each successor state is consistent before adding it to the queue.



Backtracking search

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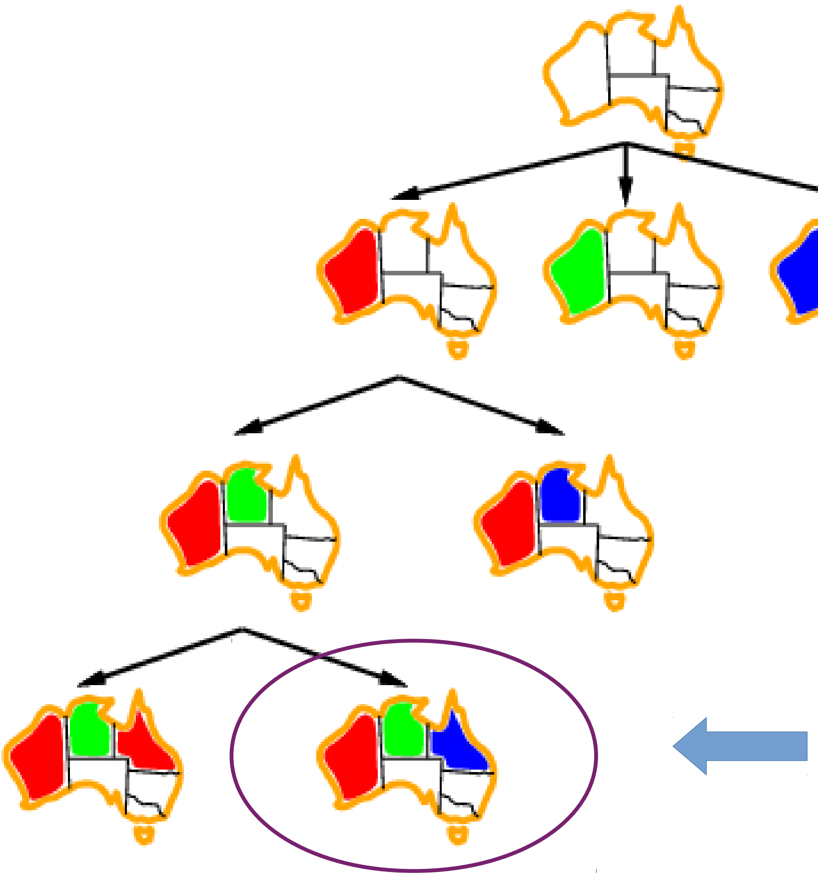

Does this state have any valid successors?

Backtracking search

```
function BACKTRACKING-SEARCH(csp) returns a solution, or failure
   return BACKTRACK({ }, csp)

function BACKTRACK(assignment, csp) returns a solution, or failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment then
         add {var = value} to assignment
         inferences ← INFERENCE(csp, var, value)
         if inferences ≠ failure then
            add inferences to assignment
            result ← BACKTRACK(assignment, csp)
            if result ≠ failure then
               return result
      remove {var = value} and inferences from assignment
   return failure
```

– backtracking enables us the ability to solve a problem as big as 25-queens

Forward checking

Sometimes, failure is inevitable:

Can we detect this situation in advance?

Forward checking

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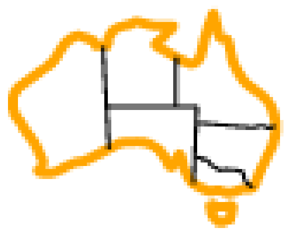
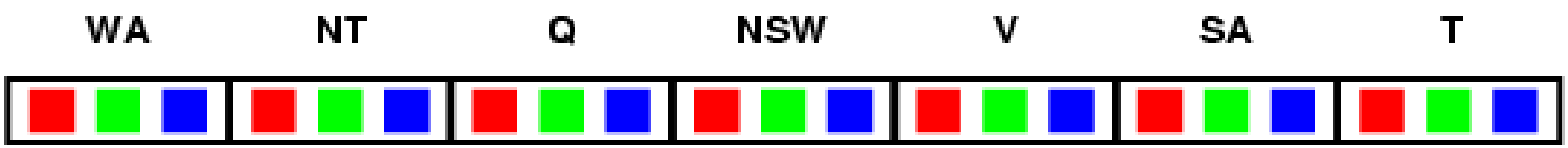
Can we detect this situation in advance?

Yes: keep track of viable variable assignments as you go

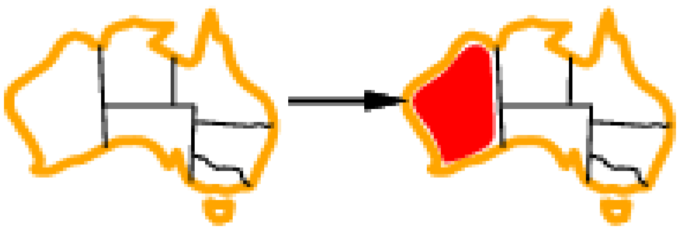
Forward checking

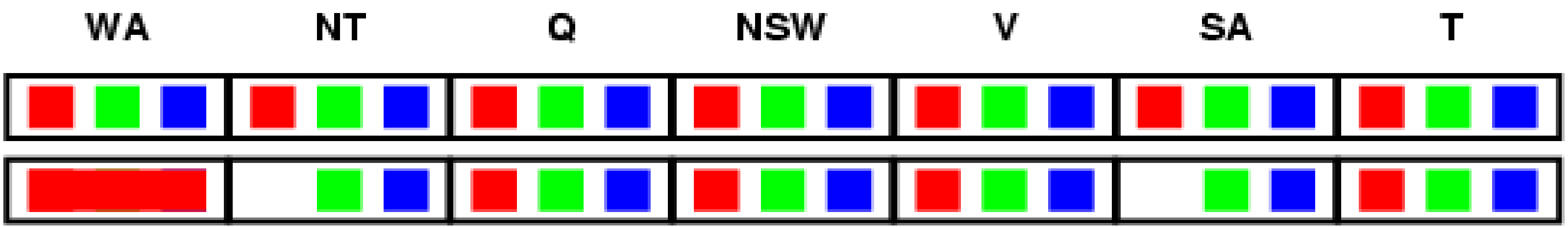




Track domain for each unassigned variable
- initialize w/ domains from problem statement
- each time you expand a node, update domains of all unassigned variables

Forward checking





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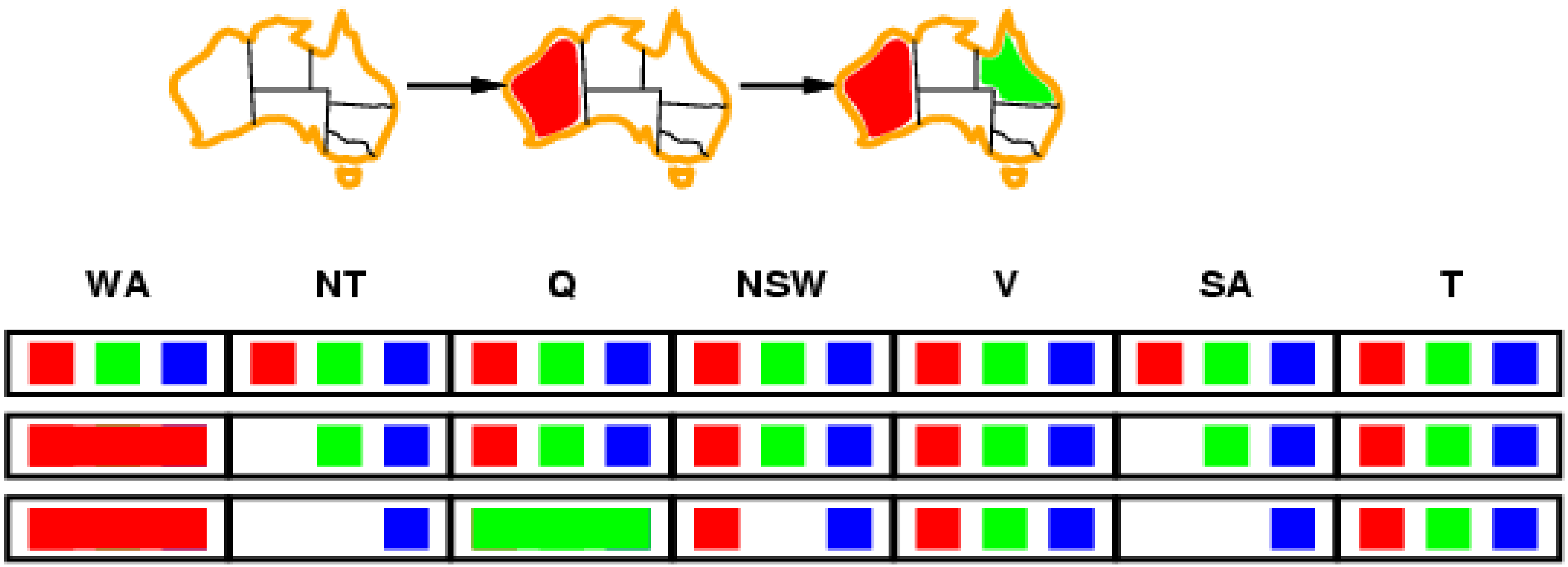

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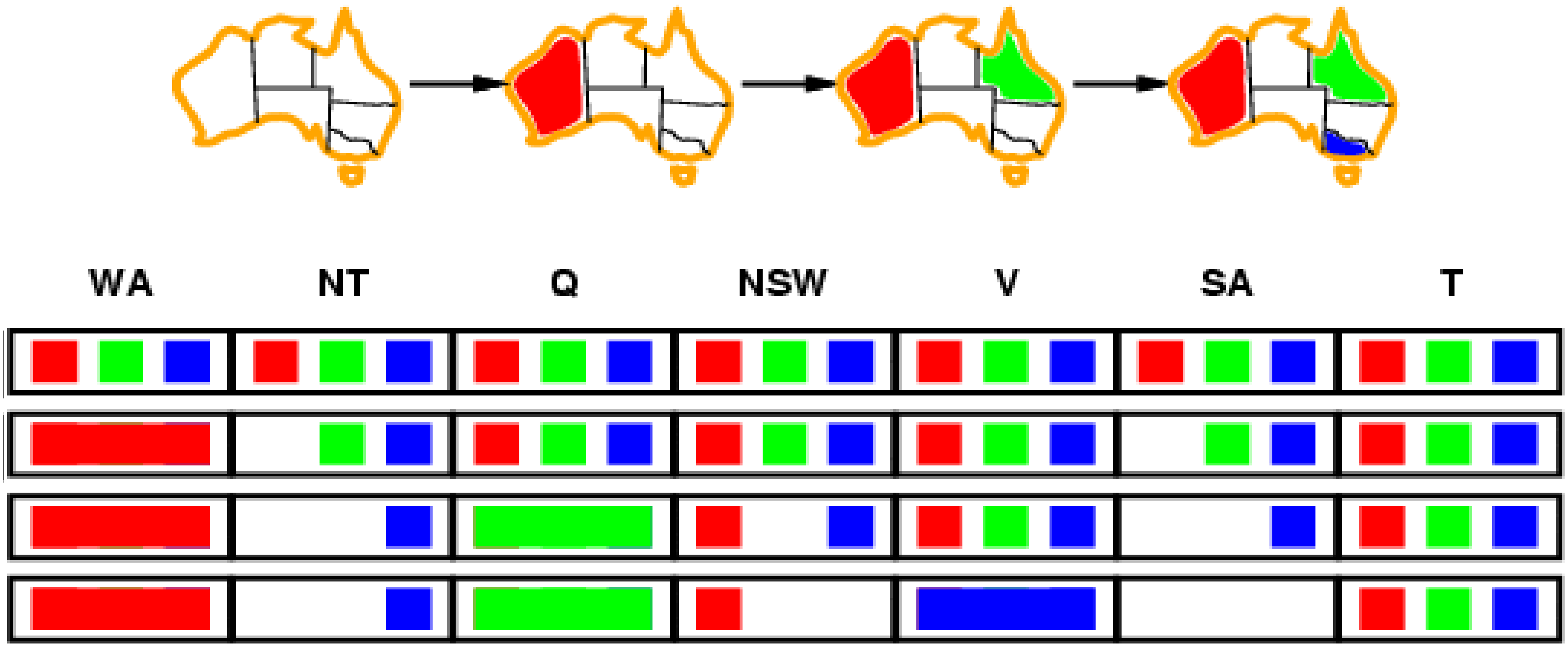

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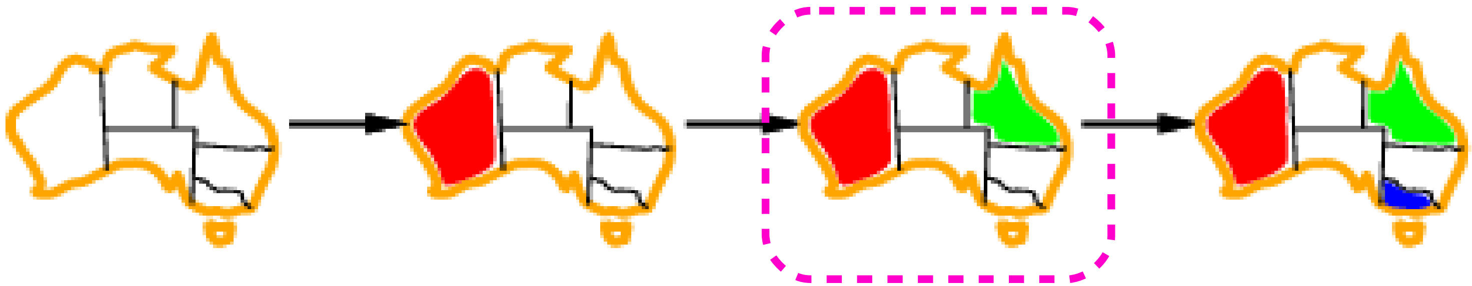
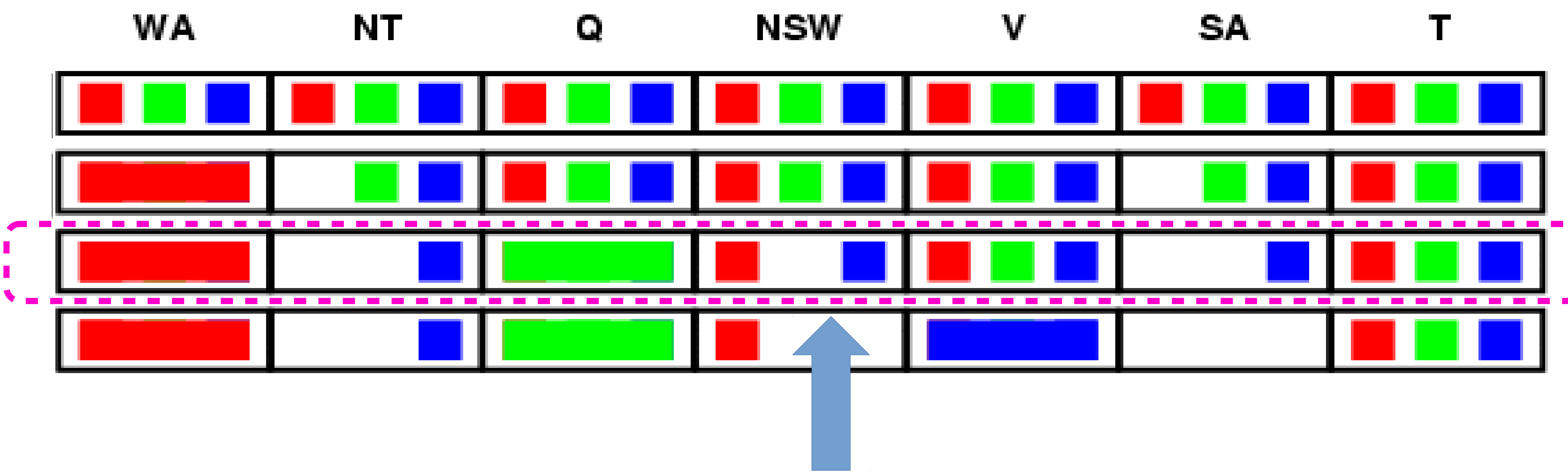
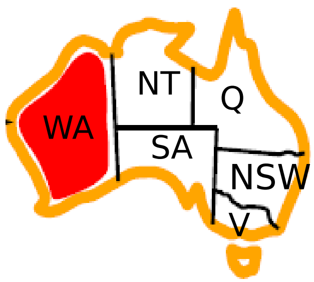


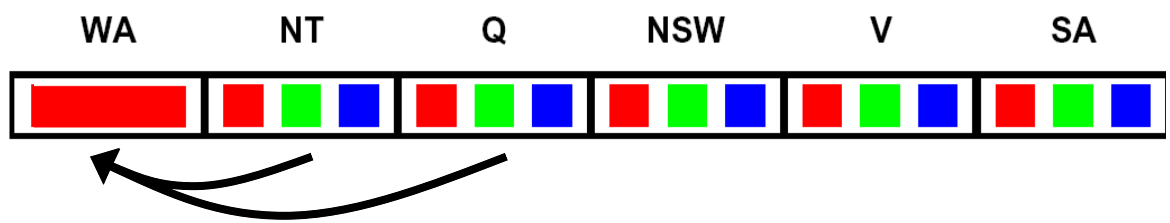


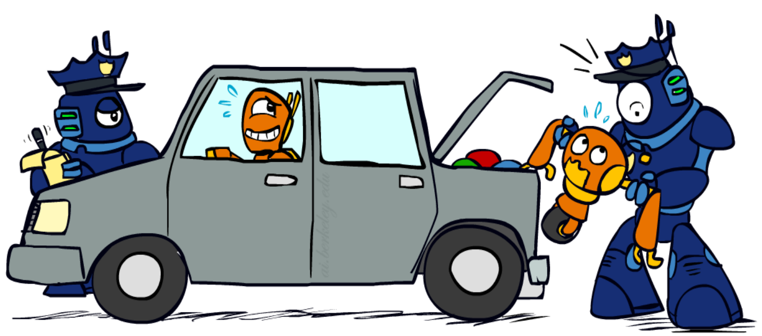
But, failure was inevitable here!
– what did we miss?

Arc consistency

- An arc X $\rightarrow$ Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint







Delete from the tail!

- Forward checking: Enforcing consistency of arcs pointing to each new assignment

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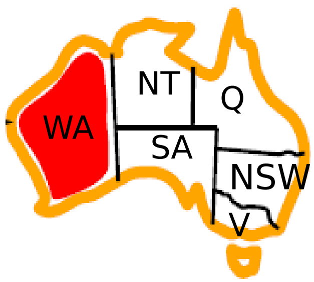

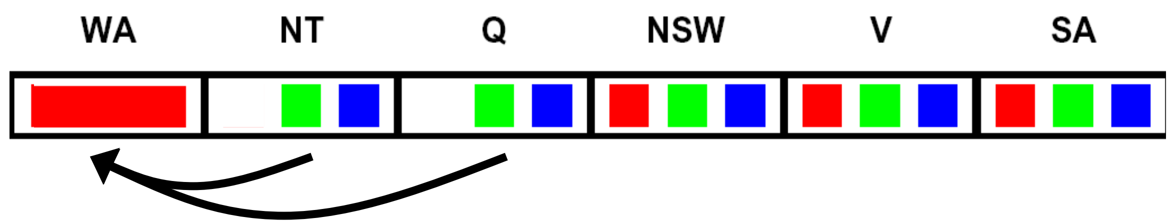

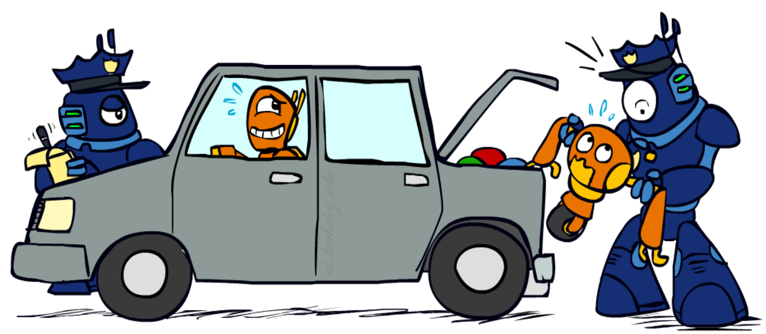

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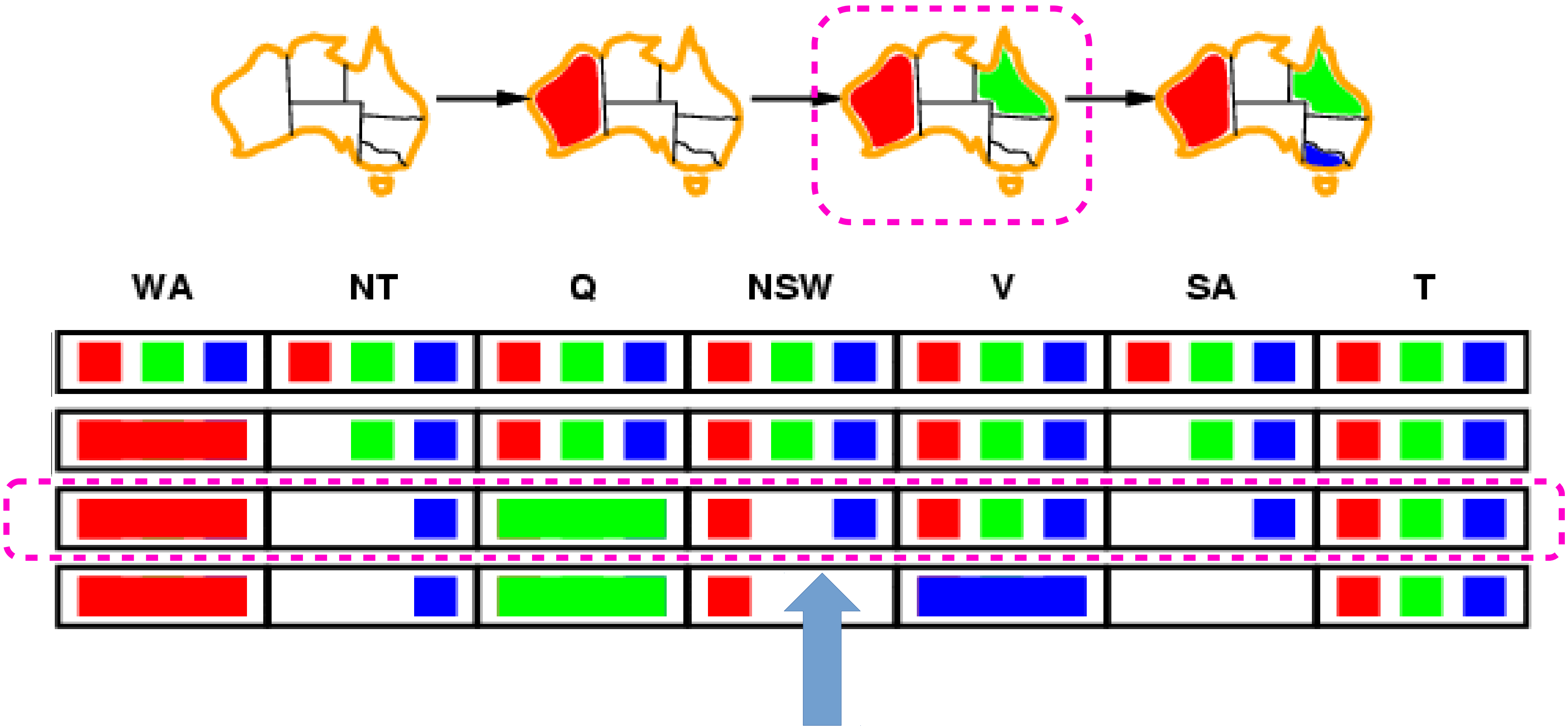

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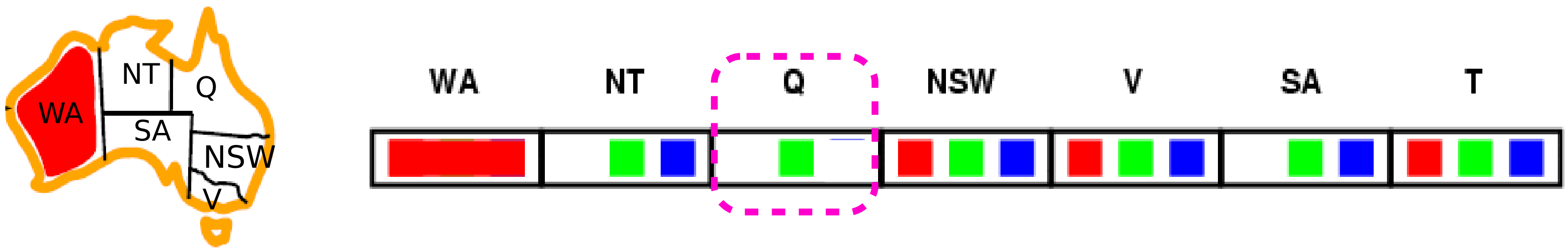
Arc consistency

- A simple form of propagation makes sure all arcs are consistent:



Delete values from tail in order to make each arc consistent

Consistent: for every value in the tail, there is some value in the head that could be assigned w/o violating a constraint.

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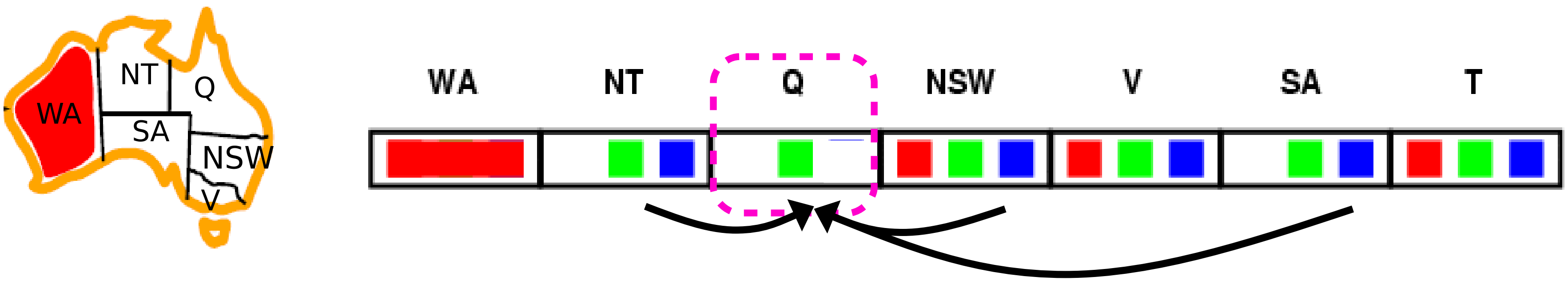

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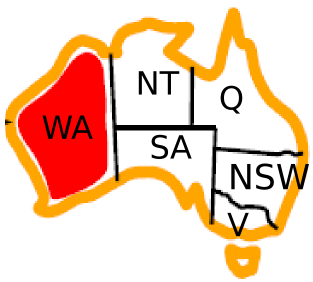

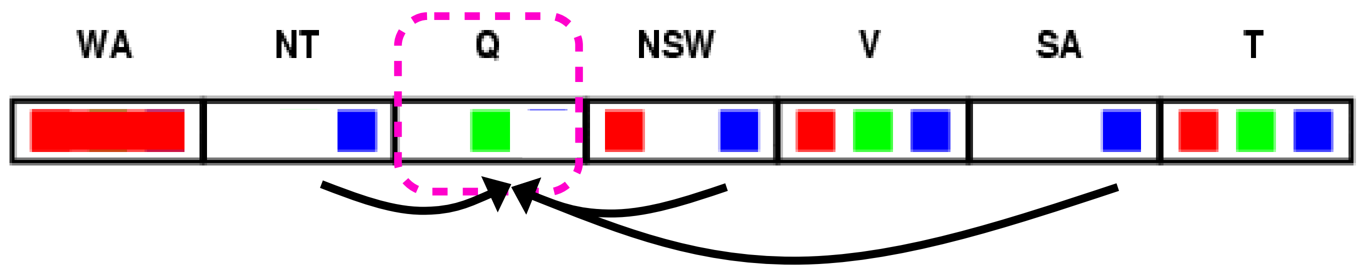

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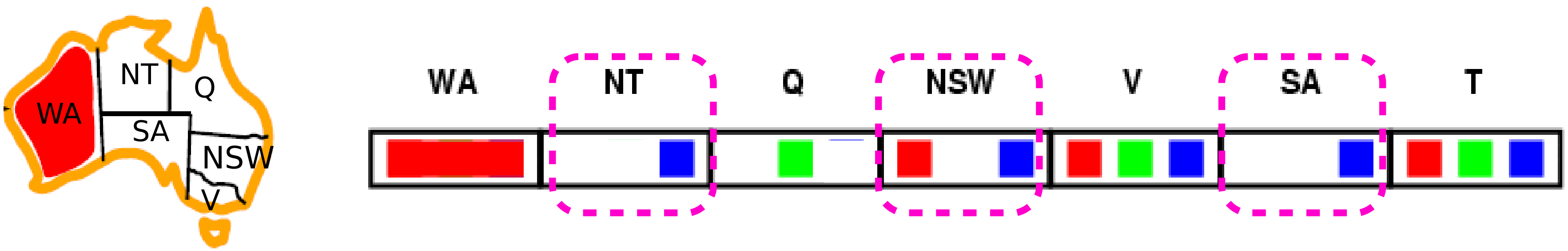

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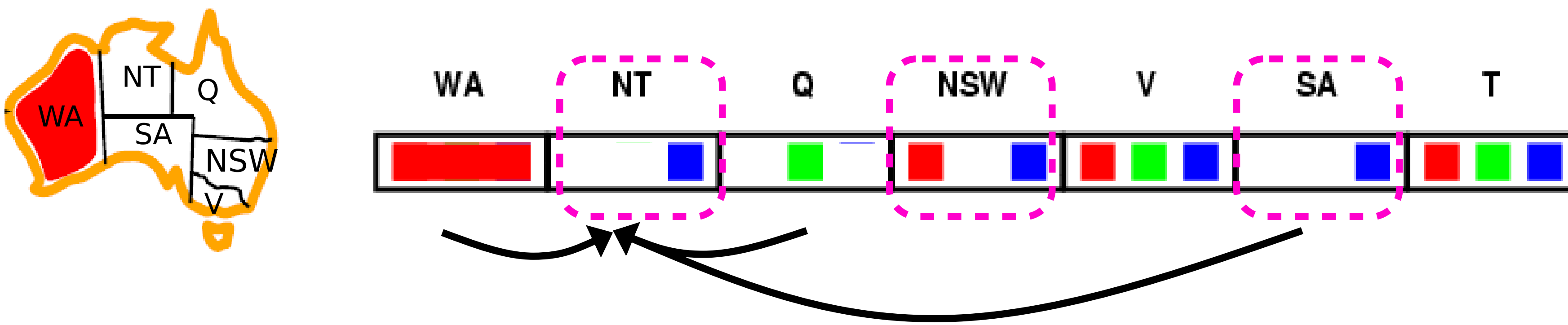

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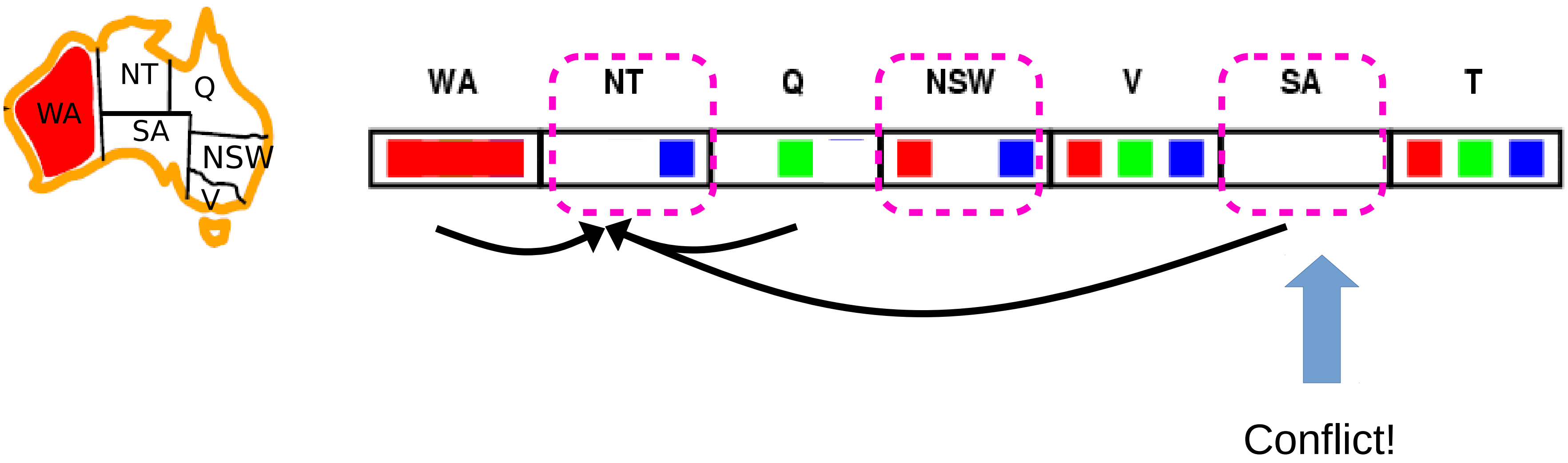

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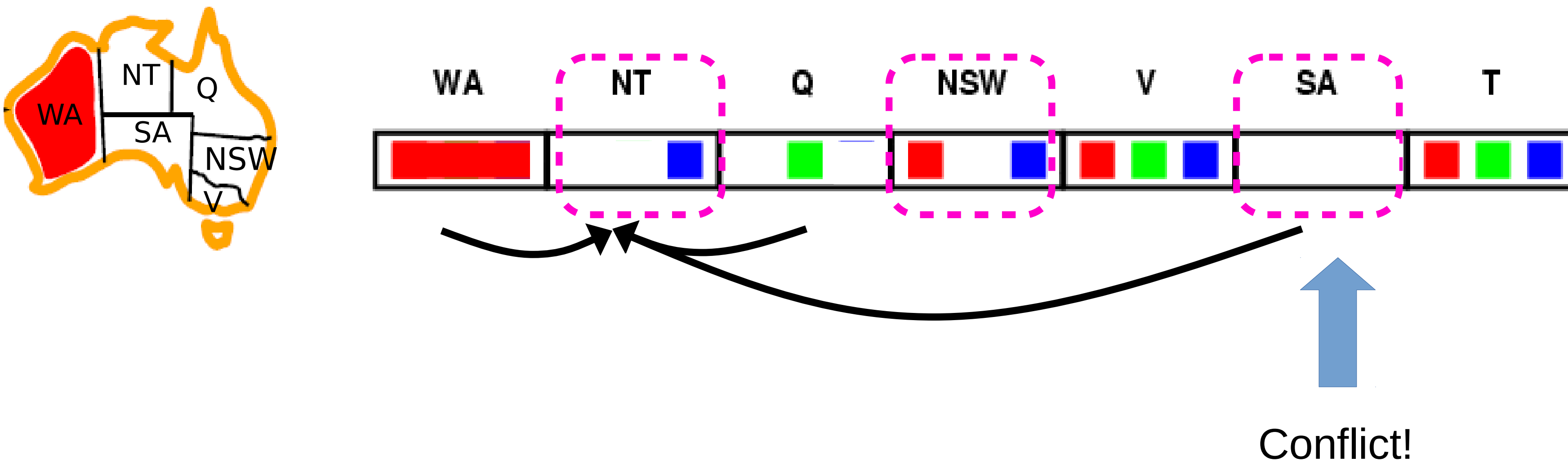


- Important: If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- What’s the downside of enforcing arc consistency?

Arc consistency

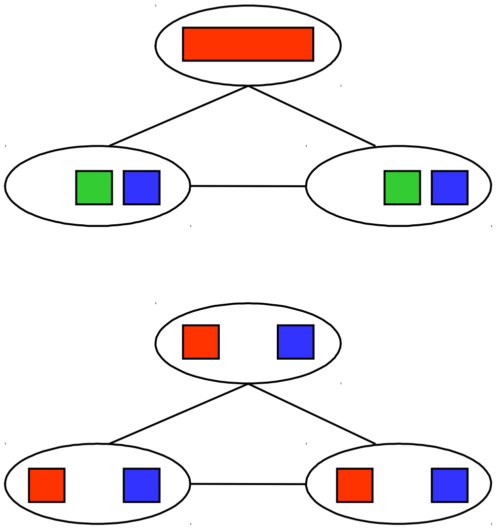
```
function AC-3( csp) returns false if an inconsistency is found and true otherwise
   inputs: csp, a binary CSP with components (X, D, C)
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (X_i, X_j) ← REMOVE-FIRST(queue)
      if REVISE(csp, X_i, X_j) then
         if size of D_i = 0 then return false
         for each X_k in X_i.NEIGHBORS - {X_j} do
            add (X_k, X_i) to queue
   return true
```

```
function REVISE( csp, X_i, X_j) returns true iff we revise the domain of X_i
   revised ← false
   for each x in D_i do
      if no value y in D_j allows (x,y) to satisfy the constraint between X_i and X_j then
         delete x from D_i
         revised ← true
   return revised
```

Why does this algorithm converge?

Arc consistency does not detect all inconsistencies...

- After enforcing arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)



What went wrong here?

- Arc consistency still runs inside a backtracking search!

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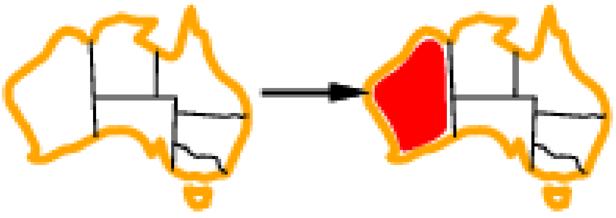
Heuristics for improving CSP performance

Minimum remaining values (MRV) heuristic:

– expand variables w/ minimum size domain first



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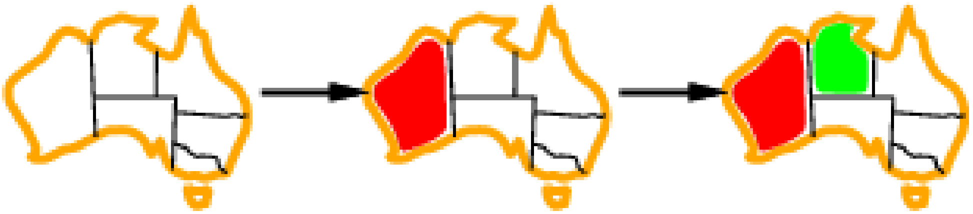

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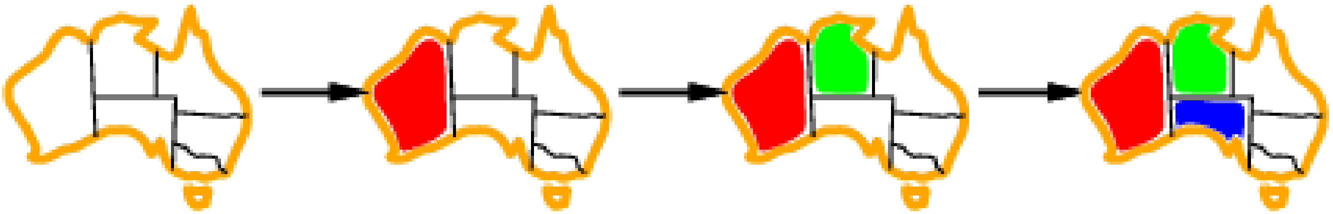
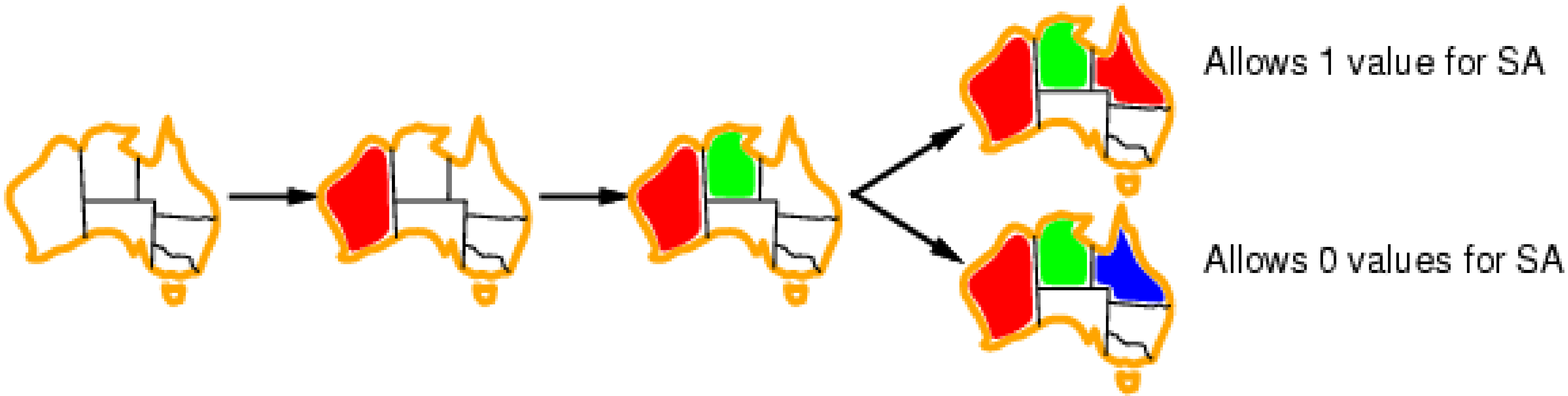


Heuristics for improving CSP performance

Least constraining value (LCV) heuristic:

– consider how domains of neighbors would change under A.C.

– choose value that contrains neighboring domains the **least**



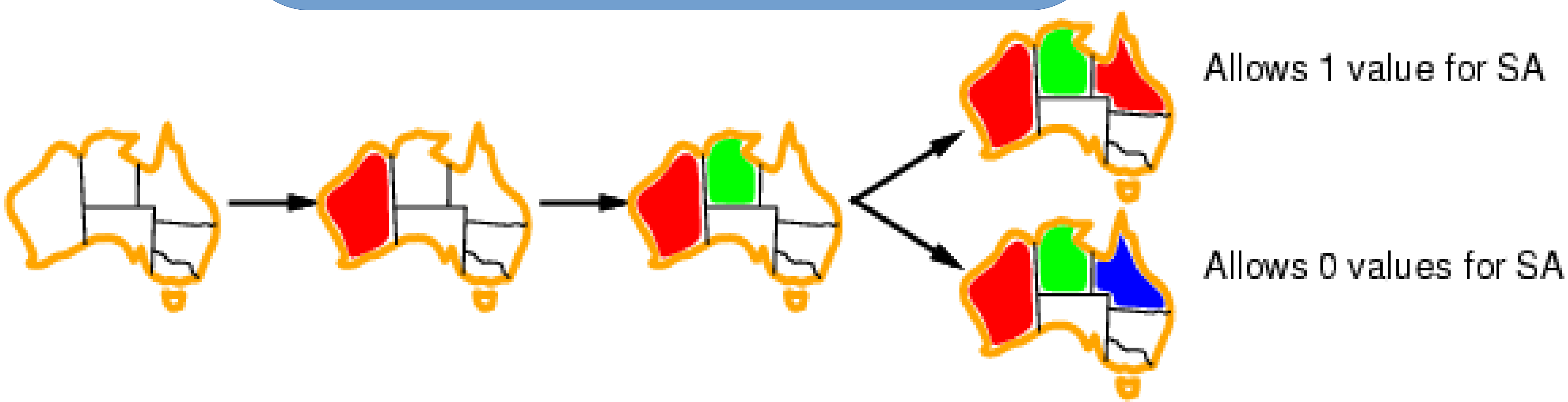
Heuristics for improving CSP performance

Least constraining value (LCV) heuristic:

– consid would change
unde

The combination of MRV and LCV w/ backtracking can solve the 1000-queens problem

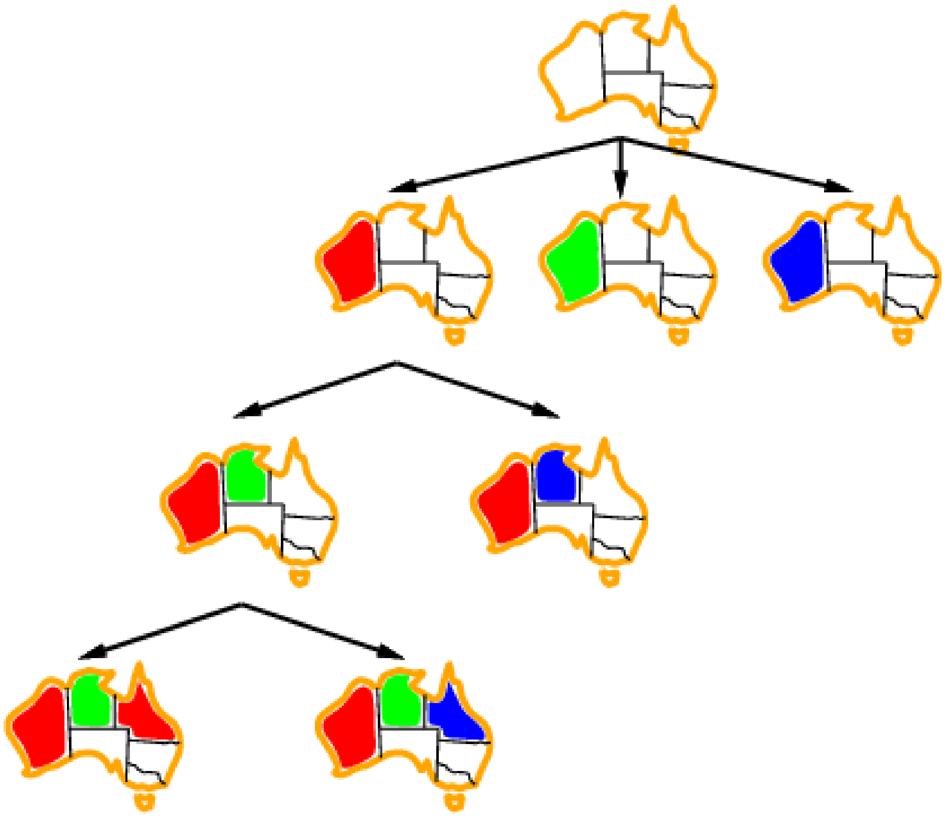
– choos ring domains
the **le**



Using structure to reduce problem complexity

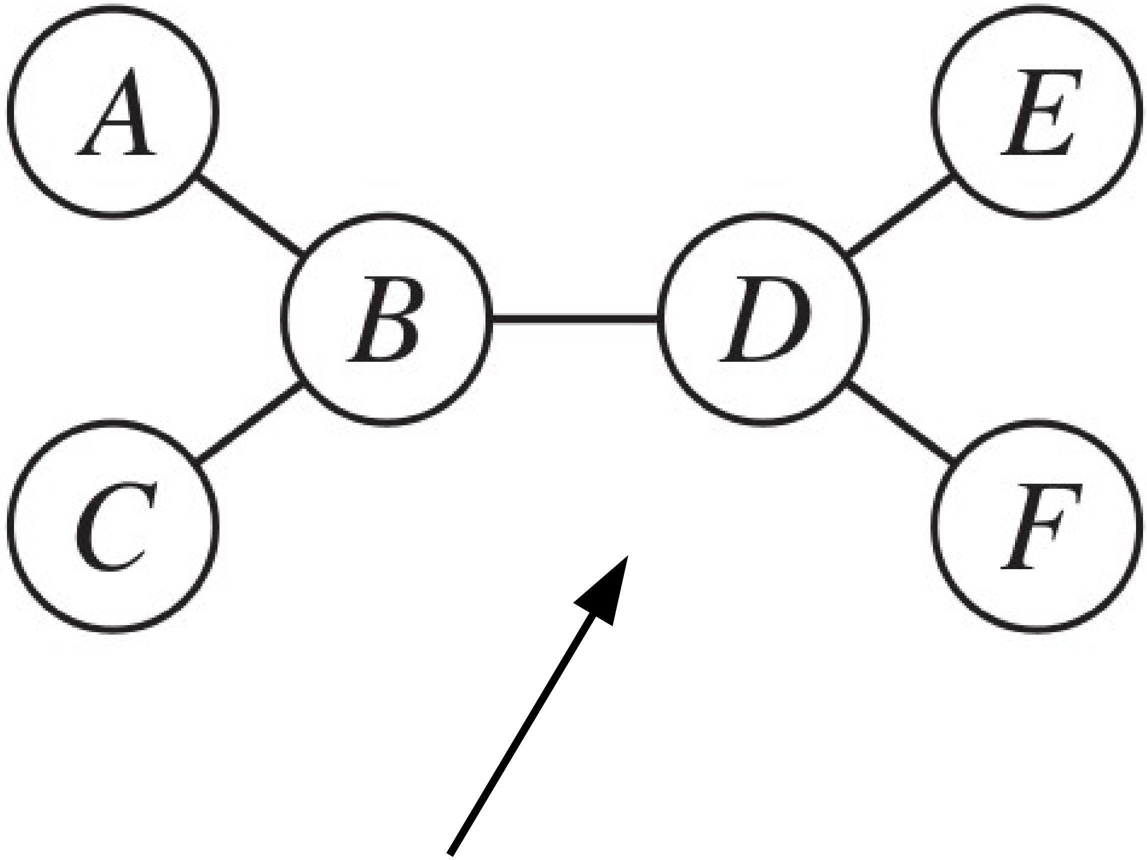
In general, what is the complexity of solving a CSP using backtracking?

(in terms of # variables, n, and max domain size, d)



But, sometimes CSPs have special structure that makes them simpler!

When the constraint graph is a tree

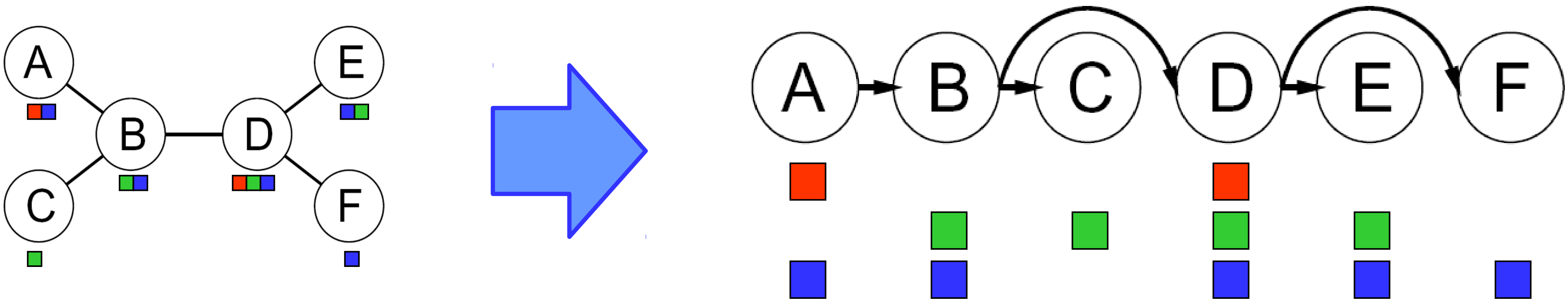


This CSP is easier to solve than the general case...

When the constraint graph is a tree

1. Do a *topological sort*
   – a partial ordering over variables

   i. choose any node as the root
   ii. list children after their parents



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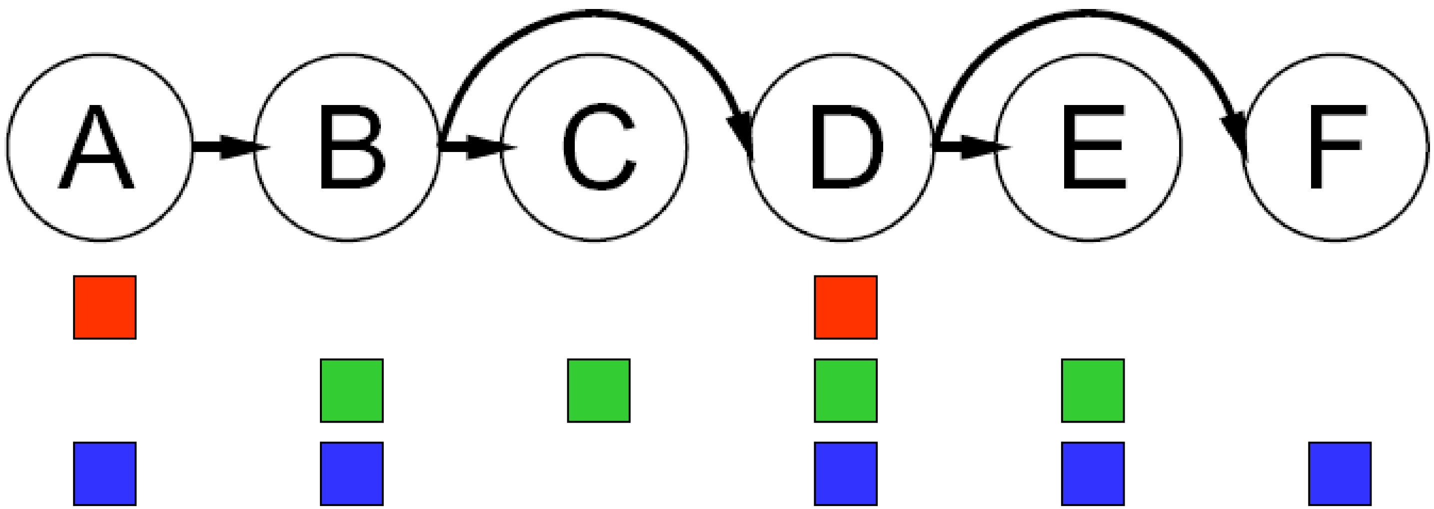
When the constraint graph is a tree

2. make the graph *directed arc consistent*
   – start w/ the tail and make each variable arc consistent wrt its parents



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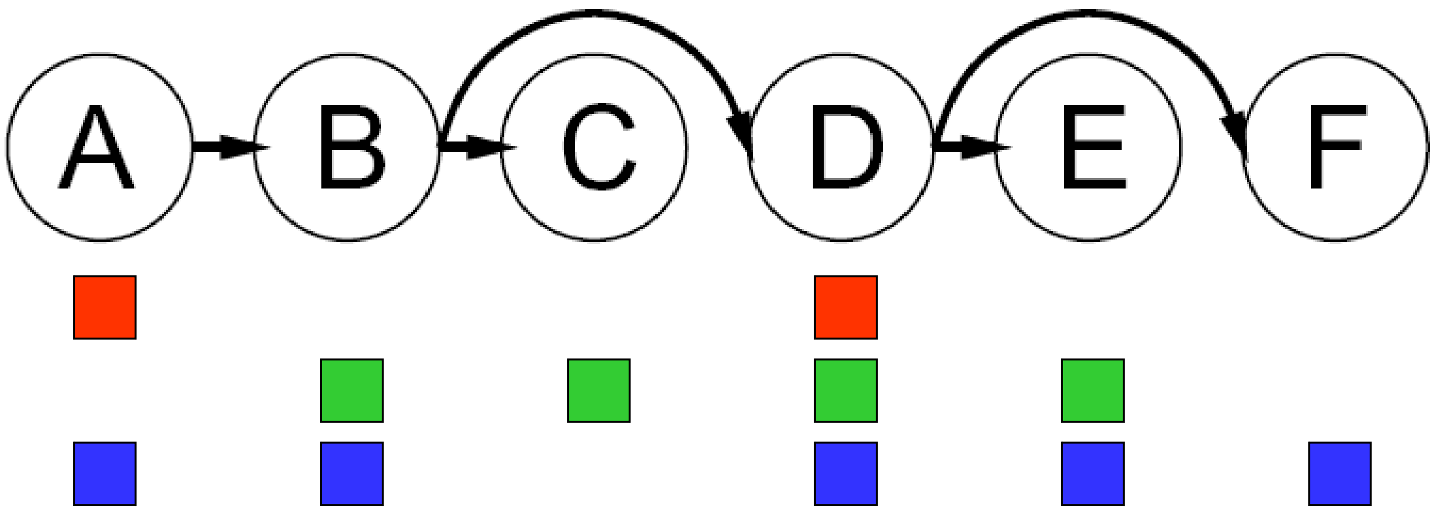

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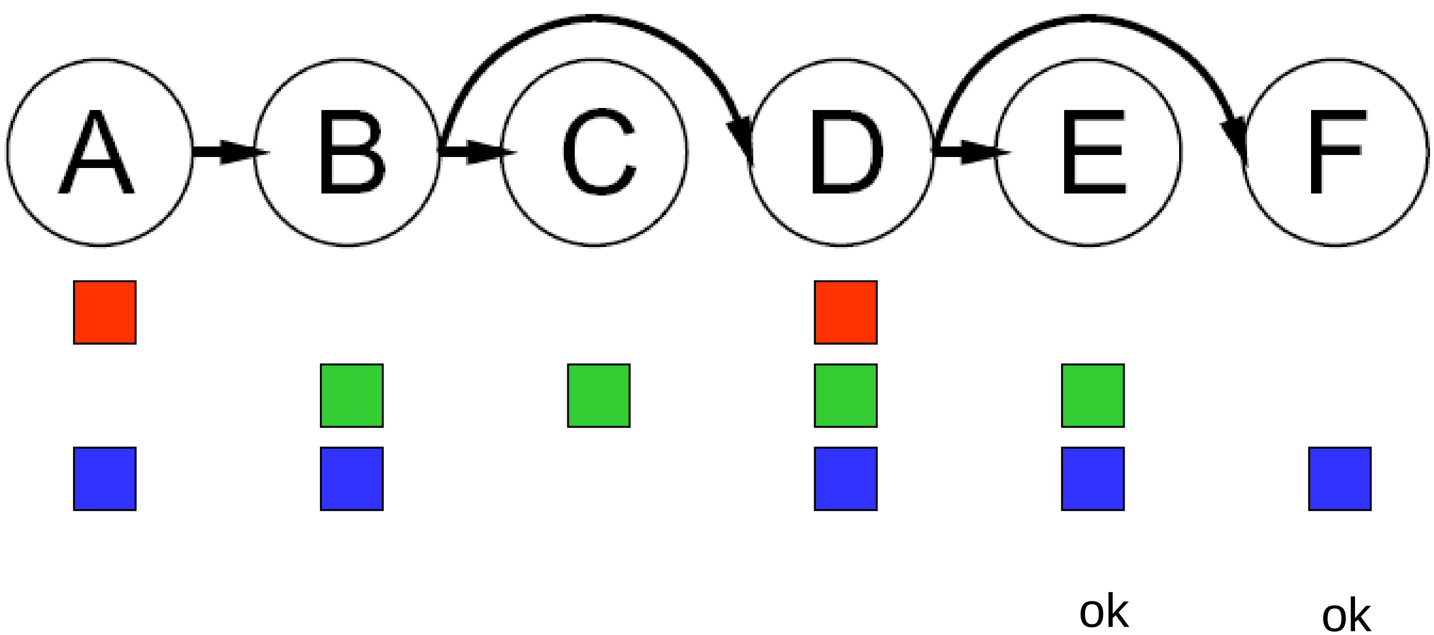

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2. make the graph *directed arc consistent*
   – start w/ the tail and make each variable arc consistent wrt its parents

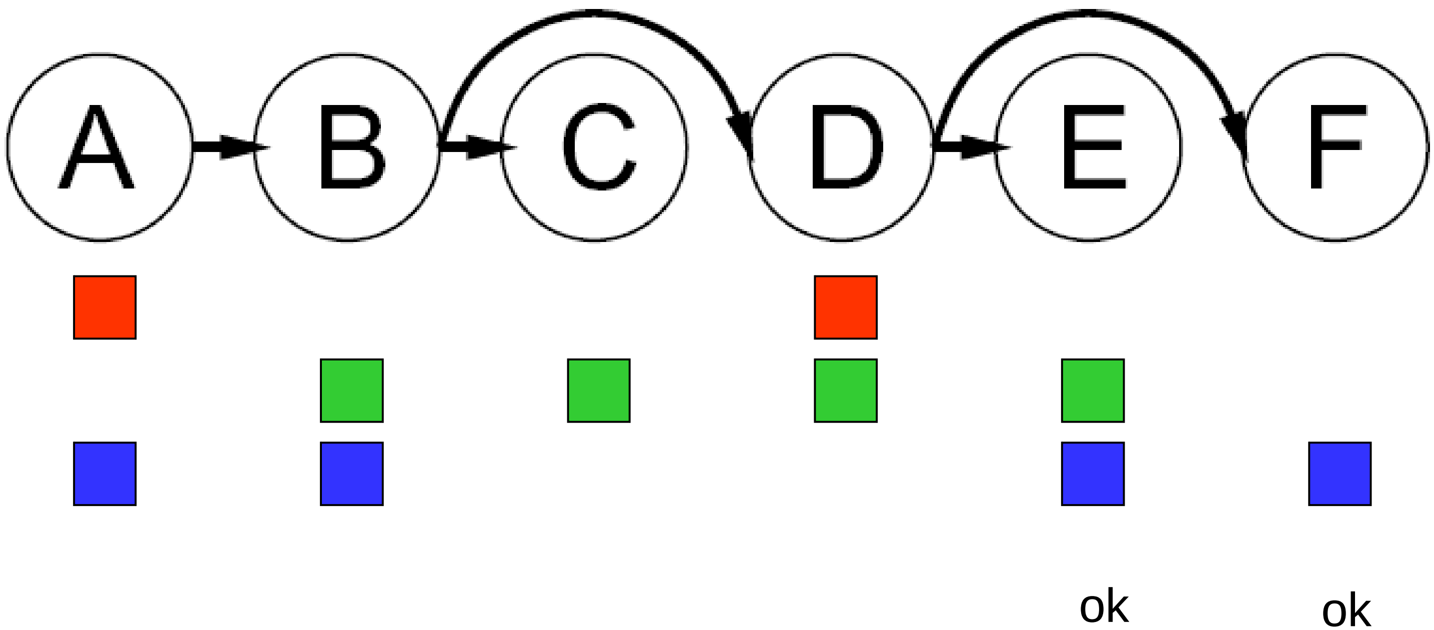

Image: Berkeley CS188 course notes (downloaded Summer 2015)

When the constraint graph is a tree

2. make the graph *directed arc consistent*
   – start w/ the tail and make each variable arc consistent wrt its parents

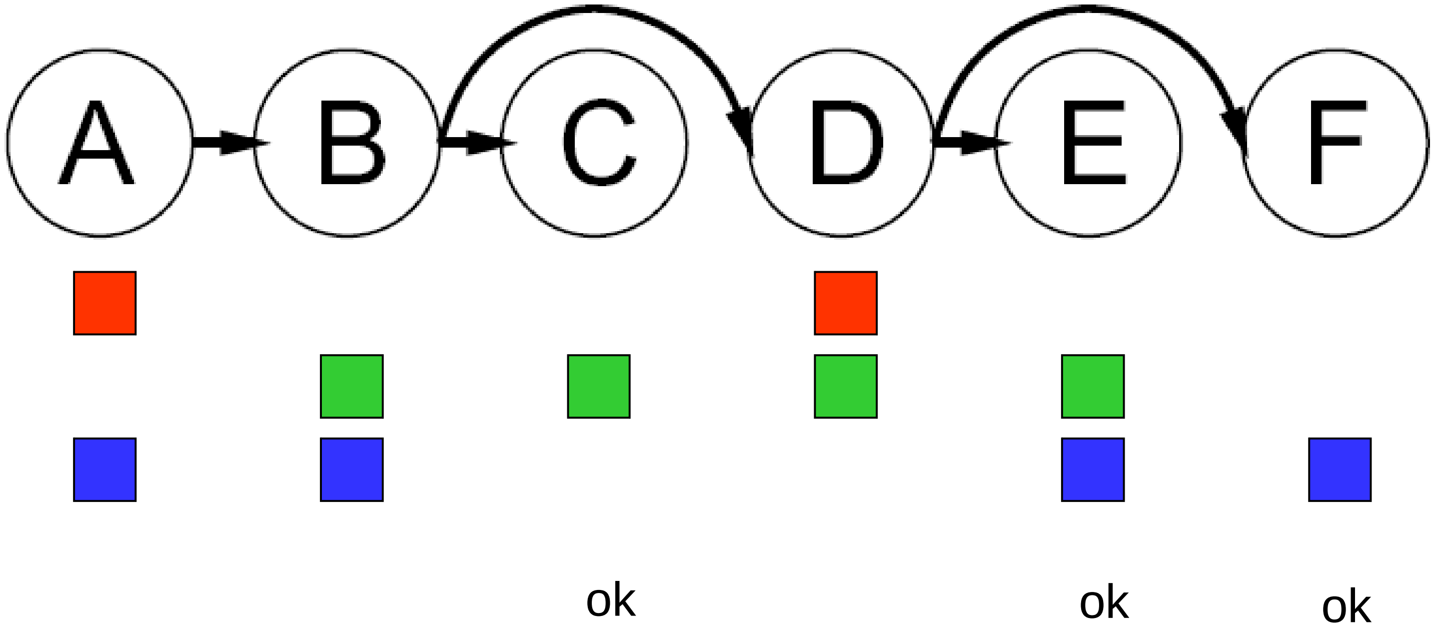

Image: Berkeley CS188 course notes (downloaded Summer 2015)

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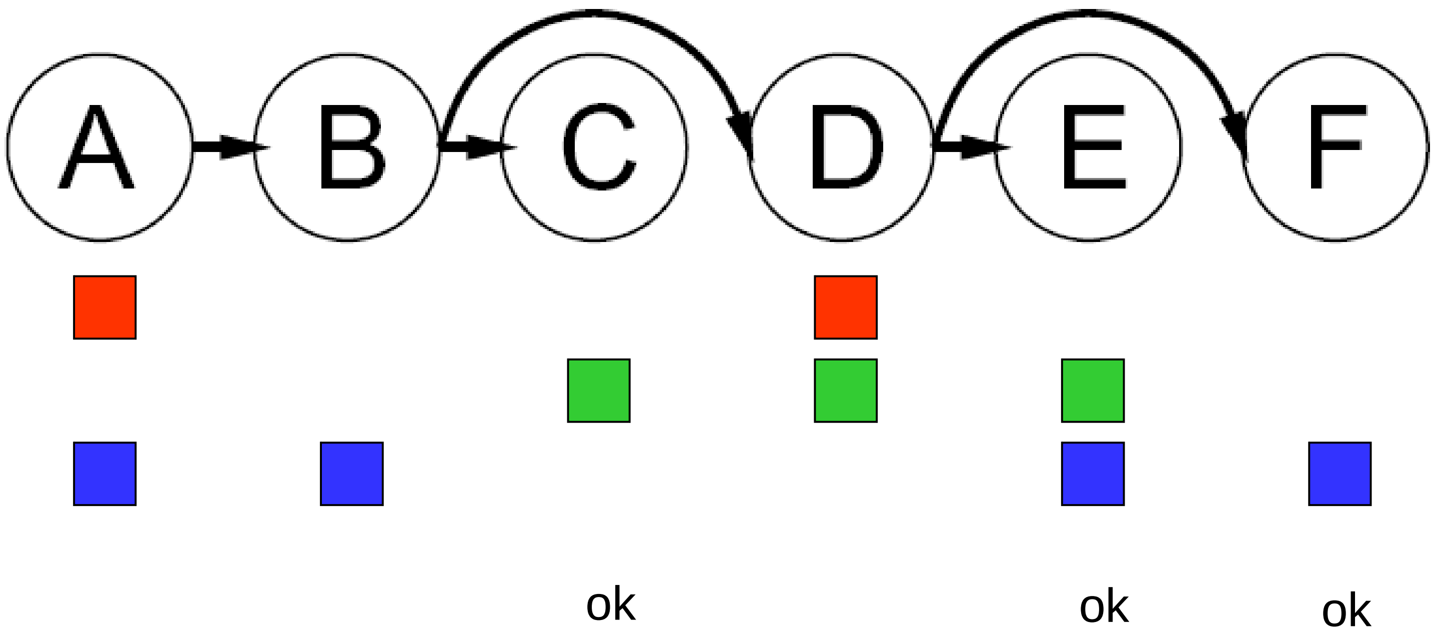

Image: Berkeley CS188 course notes (downloaded Summer 2015)

When the constraint graph is a tree

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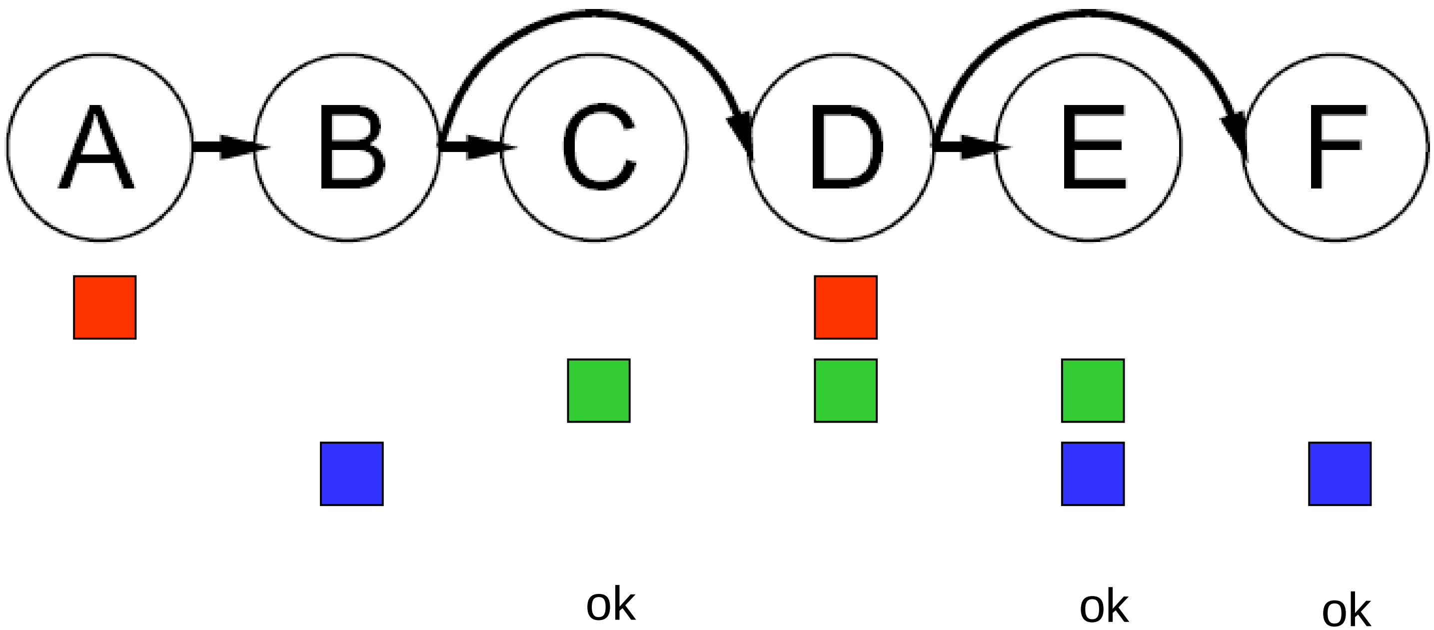
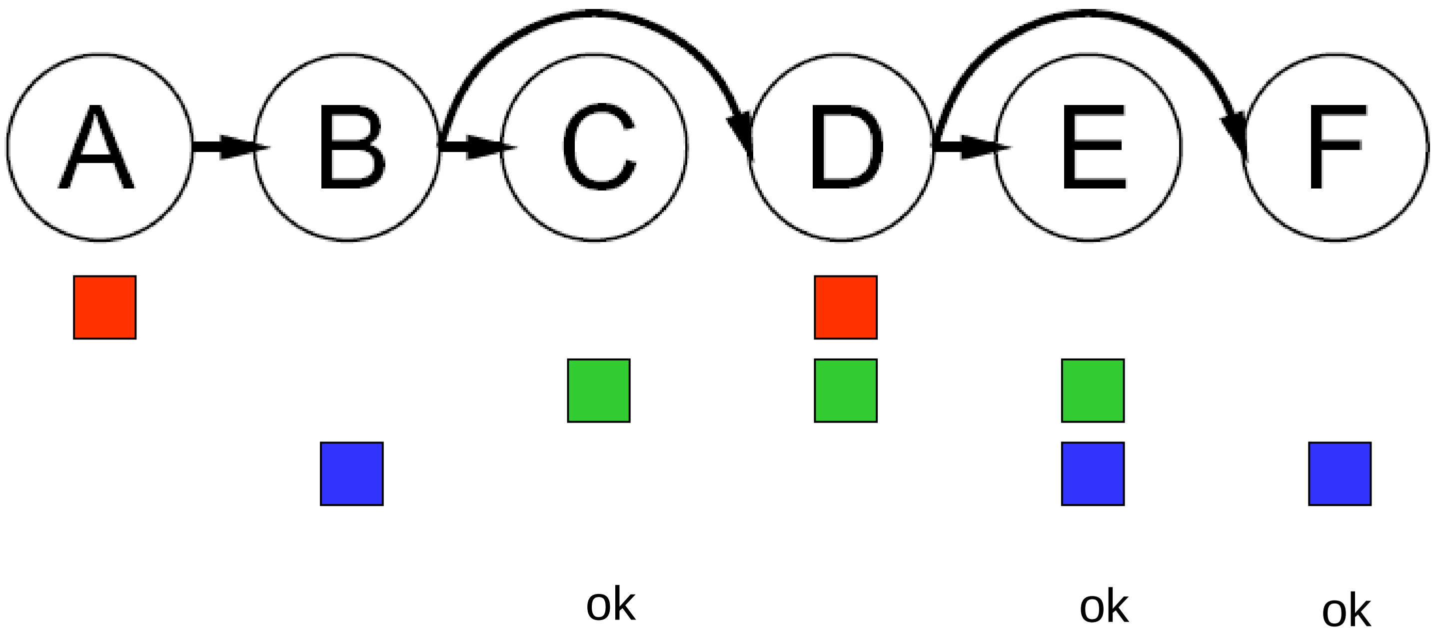


Image: Berkeley CS188 course notes (downloaded Summer 2015)

When the constraint graph is a tree

3. Now, start at the root and do backtracking
   – will backtracking ever actually backtrack?

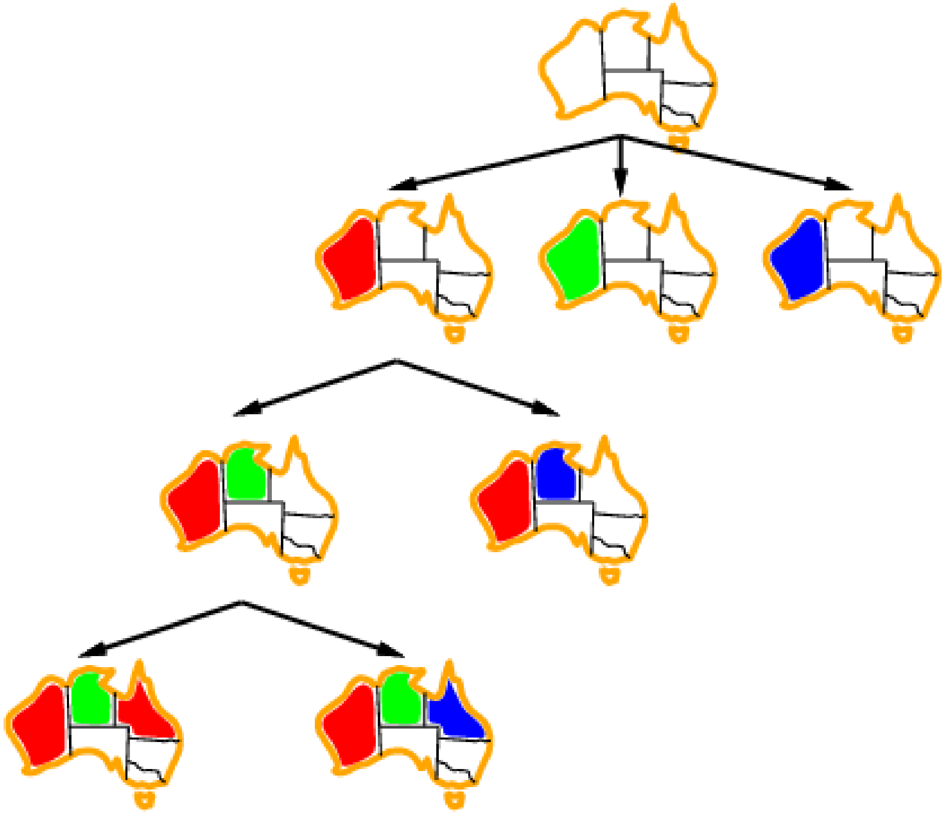


So, what's the time complexity of this algorithm?

Image: Berkeley CS188 course notes (downloaded Summer 2015)

Using structure to reduce problem complexity

But, what if the constraint graph is not a tree?
– is there anything we can do?



But, sometimes CSPs have special structure that makes them simpler!

Using structure to reduce problem complexity

But, what if the constraint graph is not a tree?
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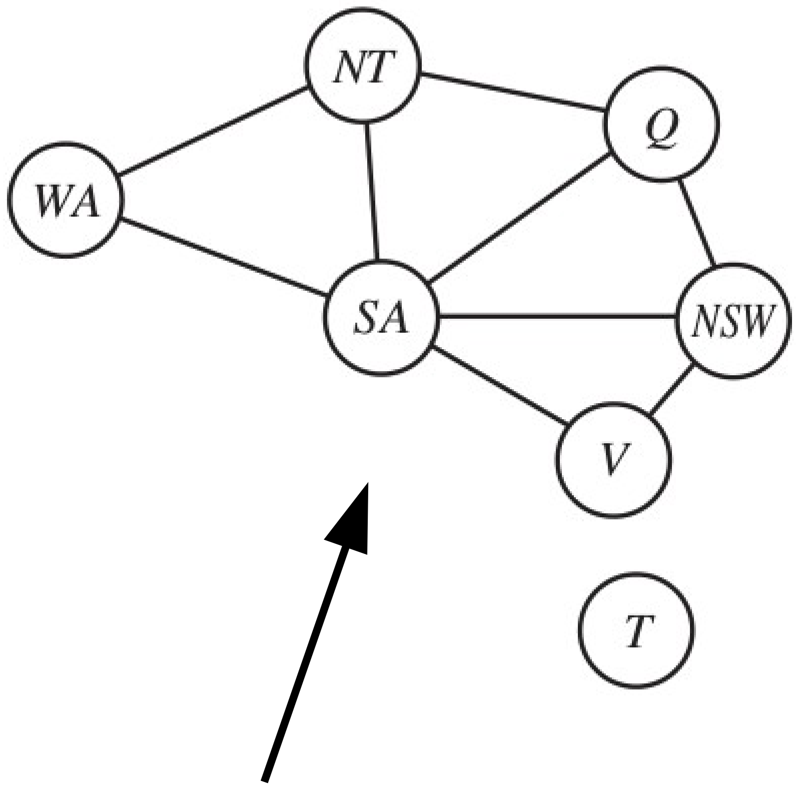
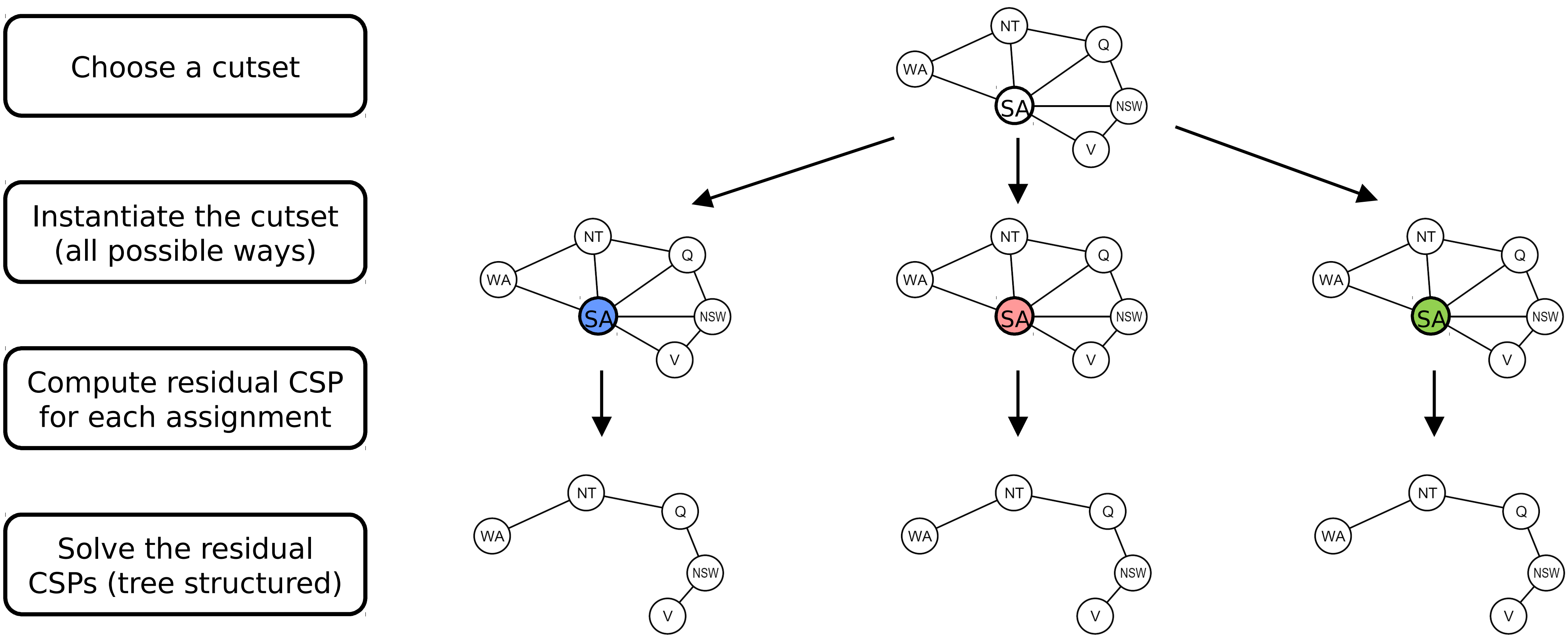


This is not a tree...

Cutset conditioning

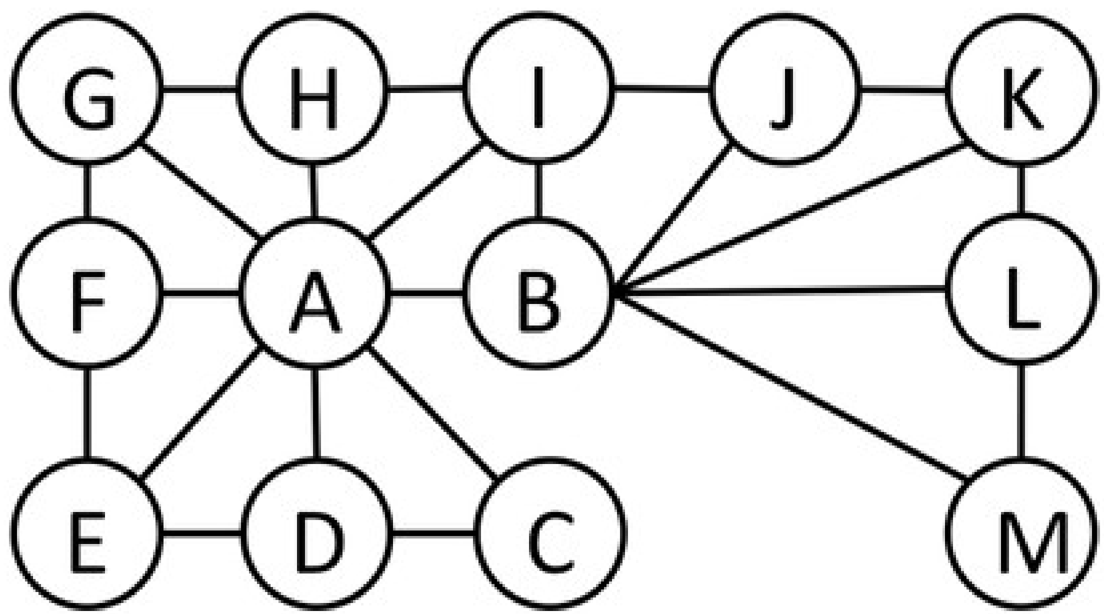


1. Turn the graph into a tree by assigning values to a subset of variables
2. For each assignment to the subset, prune domains of the rest of the variables and solve the sub-problem CSP.

– what does efficiency of this approach depend on?

Image: Berkeley CS188 course notes (downloaded Summer 2015)

Cutset conditioning



How many variables need to be assigned to turn this graph into a tree?

Image: Berkeley CS188 course notes (downloaded Summer 2015)