Homework 00

Due: Friday, September 15, 2006

Note: This review assignment will be handled somewhat differently from all subsequent assignments. In particular:

- It will not be accepted late.
- You must attempt every problem.
- Every point you earn on this assignment will count toward your homework score as extra credit points. (Please review the grading policy described in the course information page for an explanation of how extra credit points are counted.)

1. a. (5 pts) For each of these, give the resulting set by listing out all its elements:
      - $\{1,2,3\} \cap \{1,3,4,5\}$
      - $\{1,2,3\} \cup \{1,3,4,5\}$
      - $\{1,2,3\} - \{1,3,4,5\}$
      - $\{1,3,4,5\} - \{1,2,3\}$
      - $\{1,2,3\} \times \{1,3,4,5\}$

   b. (5 pts) Given any set S, the power set of S, written $\mathcal{P}(S)$ or 2^S, is the set of all subsets of S. Write out $2^{\{1,2,3\}}$.

2. a. (5 pts) How many elements did you find in $2^{\{1,2,3\}}$? How many elements are there in $2^{\{1,2\}}$? How many elements are there in $2^{\{1,2,3,4\}}$? (It is not necessary to list all of them.) In general, if S is a finite set containing n elements (which we write as $|S| = n$), make a reasonable conjecture based on these examples for a formula for $|2^S|$ in terms of n.

   b. (5 pts) Give a rigorous proof that your formula is correct for any $n \geq 0$. Hint: When creating a subset of S, for each element there is exactly one of two possibilities: it is either in this this particular subset or it is not. Use this together with the product rule for counting the overall number of combinations when multiple options are possible. In particular, the product rule says that if there are k_i options for selecting the i^{th} item and each item may be selected independently of all other items, there are $k_1 k_2 \ldots k_n$ ways of selecting a combination of all n items.

There are essentially two forms of notation we use to describe infinite sets in this class:

- using ellipses (i.e.,...); or
- using set-builder notation.

Here are two examples, described using ellipses:

- $\mathcal{N}$ = the set of all natural numbers = $\{0, 1, 2, 3, \ldots\}$; and
- $\mathcal{Z}$ = the set of all integers = $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

Here is another example, which we define using both methods: The set of all natural numbers that are perfect squares is

$$\{0, 1, 4, 9, 16, 25, \ldots\} = \{n \mid n = m^2 \text{ for some } m \in \mathcal{N}\} = \{n^2 \mid n \in \mathcal{N}\}$$

Note that whenever a set is infinite, only set-builder notation gives a mathematically rigorous specification of that set. If a set is infinite (or even finite but has more elements than we want to list out), the use of ellipses is simply a convenience designed to help our intuitive understanding, but it is not as mathematically precise as set-builder notation.

3. (5 pts) Define the set $\mathcal{N}^{\text{odd}}$ of all odd natural numbers using both methods. (Your definition using set-builder notation should refer only to $\mathcal{N}$ and should not use properties like *odd* or *even*. Hint: A natural number is odd if and only if it can be written as $2n + 1$ for some natural number n.)

4. A set S is said to be *closed* under an operation if the result of applying that operation to one or more elements of that set is always in the set. (How many elements the operation is applied to depends on how many operands that operation takes.)

   a. (5 pts) Is $\mathcal{N}$ closed under addition? Is it closed under subtraction? Explain briefly (no rigorous proof required).

   b. Prove or disprove (rigorously):

      - (5 pts) $\mathcal{N}^{\text{odd}}$ is closed under addition.
      - (5 pts) $\mathcal{N}^{\text{odd}}$ is closed under multiplication.

5. (10 pts) Recall from Problem 3 that a natural number n is odd if and only if $n = 2m + 1$ for some natural number m. It is also true that a natural number n is even if and only if $n = 2m$ for some natural number m. Give a rigorous proof that, for any natural number n, n is odd if and only if $n - 1$ is even.

6. (10 pts) Give a rigorous proof that there is no natural number l such that $l \geq n$ for all $n \in \mathcal{N}$. I.e., prove that there is no largest natural number. You may use the fact that $n + 1 > n$ for any number n.

7. (10 pts) The nation of Automobilia is famous for having hundreds of automobile manufacturers. A government official of this nation proudly claims that at least one of their automobile manufacturers has provided, for each model they make with back seats, at least one cup holder within reach of one or more back-seat passengers.

   a. A disgruntled citizen of Automobilia asserts that this claim isn't true because he can point to one particular Automobilian automobile manufacturer, Umota, whose Zeta model has a back seat but no cup holders within reach of any back-seat passengers. Does this logic refute the claim? Explain clearly why or why not.

   b. Another disgruntled citizen asserts that this claim isn't true because she knows that every automobile manufacturer in Automobilia manufactures at least one model with no back seat. Does this logic refute the claim? Explain clearly why or why not.

   c. If neither of these arguments refutes the government official's claim, explain exactly what needs to be done to prove that the claim is false.

8. (10 pts) Specialty Processors Inc. has invented a special-purpose computer chip, the Hexium, designed with only a limited set of capabilities. There are some things it cannot do at all. For example, it is known that it is not possible to write a program for a Hexium to purichow an arbitrary framboolik.

   a. Algorithm designer Al G. Rithm is interested in programs to quinziflub arbitrary gazorninplatzes. He has shown how to create such programs using a framboolik-purichowing program as a subroutine. Does it follow that it is not possible to write a program for a Hexium that quinziflubs an arbitrary gazorninplatz? Explain clearly why or why not.

b. Another algorithm designer, Dee Veloper, is interested in programs to raspirgate arbitrary halimos. She has shown that, given any program that can do this, such a program can be called with certain particular arguments (on any processor) to purichow an arbitrary framboolik. What does this imply about the possibility of writing a program on a Hexium that raspirgates an arbitrary halimo. Prove your answer.

Hint: There are three main statements to consider the truth or falseness of and/or relations between:

- A Hexium can be programmed to purichow an arbitrary framboolik.
- A Hexium can be programmed to quinziflub an arbitrary gazorninplatz.
- A Hexium can be programmed to raspirgate an arbitrary halimo.