An idempotent semiring is a structure $S = (S, +, \cdot, 1, 0)$ satisfying:

\begin{align*}
a + (b + c) &= (a + b) + c \\
a + b &= b + a \\
a + a &= a \\
a + 0 &= 0 + a = a \\
a \cdot (b \cdot c) &= (a \cdot b) \cdot c \\
a \cdot 1 &= 1 \cdot a = a \\
0 \cdot a &= a \cdot 0 = 0 \\
a \cdot (b + c) &= a \cdot b + a \cdot c \\
(b + c) \cdot a &= b \cdot a + c \cdot a
\end{align*}
They Are Not Rings!

The theory distinct from that of rings

- if $R$ is an idempotent semiring such that every element has an additive inverse, then $a = 0$ for all $a$

$$a = a + 0 = a + a + a^{-1} = a + a^{-1} = 0$$
Some Properties

Conventions

- $ab$ for $a \cdot b$
- $a^n = a \cdots \cdots a$ ($n$ times)

Well-defined partial order $\leq$ induced by $+$

- $x \leq y$ iff $x + y = y$
- $+$ and $\cdot$ monotone with respect to $\leq$
Relational Semirings

Let $X$ be a set

The set of all binary relations on $X$ forms an idempotent semiring under:

$R + S \triangleq R \cup S$

$R \cdot S \triangleq S \circ R = \{(x, z) \mid \exists y \in X. (x, y) \in R, (y, z) \in S\}$

$0 \triangleq \emptyset$

$1 \triangleq id_X$

The induced ordering is just subset inclusion
A *Kleene algebra* is a structure \( \mathcal{K} = (K, +, \cdot, *, 0, 1) \) where

- \( (K, +, \cdot, 0, 1) \) is an idempotent semiring
- The * operation satisfies

\[
1 + xx^* \leq x^* \\
1 + x^*x \leq x^* \\
b + ax \leq x \implies a^*b \leq x \\
b + xa \leq x \implies ba^* \leq x
\]

The axioms say that \( a^*b \) is the unique least solution to \( b + ax \leq x \)

- Captures the notion of *iteration*
Relational semiring are Kleene algebras under:

\[ R^* \triangleq \bigcup_{n \geq 0} R^n \]

In other words, \( R^* \) is the reflexive transitive closure of \( R \).
Every Boolean algebra \( B = (B, \wedge, \vee, -, 0, 1) \) is a Kleene algebra under

\[
\begin{align*}
a + b & \triangleq a \lor b \\
a \cdot b & \triangleq a \land b \\
a^* & \triangleq 1 \\
0 & \triangleq 0 \\
1 & \triangleq 1
\end{align*}
\]
Formal Languages

Fix a set $\Sigma$ of symbols

A *string* is a finite sequence of symbols from $\Sigma$

A *language* is a set of strings

The set of languages forms a Kleene algebra under

$$A + B \triangleq A \cup B$$

$$A \cdot B \triangleq \{xy \mid x \in A, y \in B\}$$

$$A^* \triangleq \bigcup_{n \geq 0} A^n = \{x_1 \ldots x_n \mid n \geq 0, x_i \in A\}$$

$$0 \triangleq \emptyset$$

$$1 \triangleq \{\epsilon\}$$

(where $\epsilon$ is the empty string)
Formal Languages, II

Take the smallest set of languages that:
- contains all singletons \( \{x\} \) for \( x \in \Sigma \)
- contains \( \emptyset \) and \( \{\epsilon\} \)
- is closed under +, , *

This is the \textit{free Kleene algebra} generated by \( \Sigma \)

These are exactly the regular sets over \( \Sigma \)
- Sets of strings recognizable by finite state machines
Not all Kleene algebras are isomorphic to relational Kleene algebras.

Take $\mathbb{R}_+$, the nonnegative reals, extended with $\infty$ in the usual way. This forms a Kleene algebra under

\[
\begin{align*}
    a + b &= \min\{a, b\} \\
    a \cdot b &= a + \mathbb{R} b \\
    a^* &= 0_{\mathbb{R}} \\
    0 &= \infty \\
    1 &= 0_{\mathbb{R}}
\end{align*}
\]

Does not satisfy $a \leq 1 \implies a^2 = a$
Given a Kleene algebra $\mathcal{K}$, the set of all $n \times n$ matrices over $\mathcal{K}$ forms a Kleene algebra $\text{Mat}_n(\mathcal{K})$ under the usual matrix operations, with $^*$ given inductively as follows:

\[
\begin{bmatrix}
a \\
A & B \\
C & D
\end{bmatrix}^* \triangleq \begin{bmatrix}
a^* \\
F^* & F^*BD^* \\
D^*CF^* & D^* + D^*CF^*BD^*
\end{bmatrix}
\]

where $F = A + BD^*C$
All-pairs shortest path problem

Given a directed graph $G$ with edge weights (say, representing distances between nodes)

Want to compute the length of the shortest path between all pairs of nodes efficiently

Step 1

Construct the adjacency matrix $E$, where $E_{ij}$ is the edge distance between node $i$ and node $j$, or $\infty$ if there is no edge from $i$ to $j$
Application 1

All-pairs shortest path problem

Given a directed graph $G$ with edge weights (say, representing distances between nodes)

Want to compute the length of the shortest path between all pairs of nodes efficiently

Step 2

Consider $E$ as an element of $\text{Mat}_n(\mathbb{R}_+)$ and compute $E^*$
Application 1

**All-pairs shortest path problem**

Given a directed graph $G$ with edge weights (say, representing distances between nodes)

Want to compute the length of the shortest path between all pairs of nodes efficiently

**Step 3**

Read off result: $(E^*)_{ij}$ is the length of the shortest path from node $i$ to node $j$ ($\infty$ is no path exists)
Equational reasoning about programs

Can think of a *program* as a way to describe actions to perform

Two programs are *equivalent* if they can perform the same sequences of actions

Equivalence is useful to establish correctness of program transformations
A Simple Programming Language

Let $A$ be a set of primitive actions

Grammar of a simple nondeterministic programming language:

$$S ::= a \in A$$
$$S_1; S_2$$
$$S_1 \text{ or } S_2$$
$$S^*$$

E.g., $(a; b)^*; a$
We associate to every program the *set of execution traces* that corresponds to executing the program.

We use the fact that sets of strings form a Kleene algebra:

\[
[a] \triangleq \{a\}
\]

\[
[S_1; S_2] \triangleq [S_1] \cdot [S_2]
\]

\[
[S_1 \text{ or } S_2] \triangleq [S_1] + [S_2]
\]

\[
[S^*] \triangleq [S]^*
\]

Can equationally establish that \((a; b)^*; a\) and \(a; (b; a)^*\) are equivalent programs.
As a programming language, the previous language is limited — there are no conditionals

Consider a special class of Kleene algebras

A *Kleene algebra with tests* is a Kleene algebra such that

- there exists $B \subseteq \{x \mid x \leq 1\}$
- $B$ forms a Boolean algebra with $+, \cdot, 1, 0$ as $\lor, \land, 1, 0$
Let $\Sigma$ be a set of symbols

Let $P$ be a set of primitive tests

let $\mathcal{B}_P$ be the free Boolean algebra generated by $P$

The set of languages over $\Sigma$ and $\mathcal{B}_P$ forms a Kleene algebra with tests

- The embedded Boolean algebra consists of $\{b\}$ for $b \in \mathcal{B}_P$

- We identify $\wedge, \vee, 1, 0$ and $\cdot, +, 1, 0$
Another Programming Language

Let $A$ be a set of primitive actions
Let $P$ be a set of primitive tests

Grammar of a nondeterministic programming language:

$$T ::= p$$
$$\neg T$$
$$T_1 \land T_2$$
$$T_1 \lor T_2$$

$$S ::= a$$
$$S_1; S_2$$
$$\text{if } T \text{ then } S_1 \text{ else } S_2$$
$$\text{while } T \text{ do } S$$
Semantics

We give a semantics using sets of strings, this time relying on the fact that we have a Kleene algebra with tests

Tests $T$ get mapped to elements of the embedded Boolean algebra

\[
[p] \triangleq \{p\}
\]

\[
[-T] \triangleq \overline{[T]}
\]

\[
[T_1 \land T_2] \triangleq [T_1] \cdot [T_2]
\]

\[
[T_1 \lor T_2] \triangleq [T_1] + [T_2]
\]
We give a semantics using sets of strings, this time relying on the fact that we have a Kleene algebra with tests.

Statements get mapped to sets of strings

\[
\begin{align*}
[a] & \triangleq \{a\} \\
[S_1; S_2] & \triangleq [S_1] \cdot [S_2] \\
\text{if } T \text{ then } S_1 \text{ else } S_2 & \triangleq ([T] \cdot [S_1]) + ([T] \cdot [S_2]) \\
\text{while } T \text{ do } S & \triangleq ([T] \cdot [S])^* \cdot [T]
\end{align*}
\]
Closing Remarks

Classification results?

- Standard decomposability questions
- In how many ways can you derive a Kleene algebra from an idempotent semiring?

Representation theorems?

- Under what conditions is a Kleene algebra isomorphic to a relational Kleene algebra?
- Is there a natural representation for all Kleene algebras?