**Theorem.** A language $A$ is regular if and only if there exists an NFA $M$ such that $L(M) = A$.

**Proof.** The forward direction is trivial, since $A$ regular means there is a DFA that recognizes it, and a DFA can be seen as an NFA rather immediately. So we focus on the backward direction. Assume that $A$ is a language that is recognized by an NFA $M = (Q, \Sigma, \Delta, q_0, F)$. Without loss of generality, we know we can take the NFA to have no $\epsilon$ transitions. To show $A$ is regular, we need to construct a DFA $M' = (R, \Sigma, \delta, r_0, G)$ that recognizes $A$. (To distinguish states of $M$ from states of $M'$, we use $r$ to range over states of $M'$, and $R$ to represent the set of all states.)

The DFA $M'$ will simulate the NFA $M$, in the sense that when following symbols of a string in $M'$, the path taken will somehow capture all the possible paths that can be taken in $M$.

Define $M' = (R, \Sigma, \delta, r_0, G)$ by taking:

- $R = \{ r \mid r \subseteq Q \}$
- $\delta(r, a) = \{ q' \mid q' \in \Delta(q, a) \text{ for some } q \in r \}$
- $r_0 = \{ q_0 \}$
- $G = \{ r \mid r \cap F \neq \varnothing \}$

We now need to verify that $L(M') = L(M)$, that is, that $M'$ and $M$ recognize the same language. In other words, we need to show that for every string $w$, $M$ accepts $w$ if and only if $M'$ accepts $w$.

We do this by induction on the length of $w$ (since we need to show something true for an infinite number of things). First, define some useful notation. If $\delta$ is the transition relation of a DFA, then $\delta^*(q, w)$ tells you which state you end up in if you follow all the symbols in $w$ from state $q$, based on the transition $\delta$. Formally, $\delta^*$ is defined inductively on the structure of a string:

$$
\delta^*(q, \epsilon) = q \\
\delta^*(q, w \cdot a) = \delta(\delta^*(q, w), a),
$$

where $w \cdot a$ is the concatenation of string $w$ and symbol $a$. It is not hard to show that a DFA $M = (Q, \Sigma, \delta, q_0, F)$ accepts $w$ if and only if $\delta^*(q_0, w)$ is in $F$. Similarly, if $\Delta$ is the transition relation of an NFA without $\epsilon$ transitions, we can define $\Delta^*$ that tell you which states you can end up in if you follow
all the symbols in \( w \) from state \( q \) of the NFA, based on the transition \( \Delta \). As above:

\[
\Delta^*(q, \epsilon) = \{q\}
\]

\[
\Delta^*(q, w \cdot a) = \cup_{q' \in \Delta^*(q, w)} \Delta(q', a)
\]

Again, it is not hard to show that an NFA \( M = (Q, \Sigma, \Delta, q_0, F) \) accepts \( w \) if and only \( \Delta^*(q_0, w) \) has a state that appears in \( F \).

Now, given our automata \( M \) and \( M' \) as defined above, we show that for all strings \( w \), \( \delta^*(r_0, w) = \Delta^*(q_0, w) \). For the base case \( w = \epsilon \), since \( r_0 = \{q_0\} \), we have \( \delta^*(r_0, \epsilon) = \delta^*(\{q_0\}, \epsilon) = \{q_0\} = \Delta^*(q_0, \epsilon) \), as required.

For the inductive case, assume the result is true for a string \( w \), we need to show it is true for a string \( w \cdot a \): By definition, \( \delta^*(r_0, w \cdot a) = \delta(\delta^*(r_0, w), a) \).

By the induction hypothesis, \( \delta^*(r_0, w) = \Delta^*(q_0, w) \), and thus \( \delta^*(r_0, w \cdot a) = \delta(\Delta^*(q_0, w), a) \).

By the definition of \( \delta \), \( \delta(\Delta^*(q_0, w), a) = \cup_{q \in \Delta^*(q_0, w)} \Delta^*(q, a) \), which is just \( \Delta^*(q_0, w \cdot a) \), as required. This proves the statement.

Now, suppose that \( M \) accepts \( w \), that is, \( \Delta^*(q_0, w) \cap F \neq \emptyset \). By the above result, this is equivalent to \( \delta^*(r_0, w) \cap F \neq \emptyset \), that is, \( \delta^*(r_0, w) \in G \), and this is equivalent to \( M' \) accepting \( w \). This establishes that \( M \) and \( M' \) accept the same strings, that is, recognize the same language.