Maximal Ordinals Notations

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Recall: Ordinal Notations

- An ordinal notation for ordinal α is an explicit, constructive injection from $A \subseteq \omega$ to α
- For example, polynomials and numbers give us ω
- Add ω and we get the ordinals up to ε_0
- Add ε_0 and we can go further
- We can keep adding new symbols forever
- What about a "maximal" notation system?
- Kleene and others considered such questions

Gödel Numbering

- Recall: an ordinal notation for ordinal α is an explicit, constructive injection from $A \subseteq \omega$ to α
- For example, consider the ordinals up to ε_0 in ACL2
- Can use Gödel encodings to go from lists to ω
- Can Gödel number Turing Machines
- If e is the Gödel number of a Turing Machine then
  - {e} is the corresponding TM
  - {e}(x) is the result of running the TM on input x
- Key definition: a *fundamental sequence* for limit ordinal λ is an increasing ω-sequence of ordinals whose limit is λ

Constructive Ordinal Notation Systems

- A *constructive ordinal notation system* (CONS) is a pair (L, f) s.t. $L \subseteq \omega$, $f: L \rightarrow On$, and K,P, S are programs s.t.:
  - If f.x = 0, then K.x = 1
  - If f.x is a successor ordinal, then K.x = 2
  - If f.x is a limit ordinal K.x = 3
  - If f.x is $\alpha+1$, then P.x is a notation for α
  - If f.x is a limit ordinal λ, then S.x is the Gödel number of a program s.t. {S.x}(0), {S.x}(1), {S.x}(2), ... are notations for a fundamental sequence for λ
- Example: let 2^i denote the naturals and let $3 \cdot 2^i$ denote $\omega+i$. What are L, f? Define K, P, and S.
- Note, unique notations not required

Kleene's Ordinal Notation System

- We describe CONS (L, f) by describing the set $f^{-1}(\alpha)$. L is the union of all non-empty sets $f^{-1}(\alpha)$
  - 0 is the unique notation for 0
  - 2^x is a notation $\alpha+1$ iff x is a notation for α
  - 3^e is a notation for limit ordinal λ iff {e}.0, {e}.1, {e}.2, ..., are notations for a fundamental sequence for λ
- What notations do 0, 1, 2, 3, 4, 5, ... receive?
- 0, 1, 2, 4, 16, 2^{16}, ... And, the notations are unique
- What is a notation for ω?
- Numbers of the form 3^e, where {e} outputs an increasing subsequence of 0,1,2,4,16,2^{16}, ...
- For each such e, $2^{\wedge}(3^e)$ is a notation for $\omega+1$, and so on
- Define K, P, and S

Kleene's Ordinal Notation System

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  - 2^x is a notation $\alpha+1$ iff x is a notation for α
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- Is this a CONS? What are K, P, and S?
- Theorem: for distinct ordinals α, β in the range off, $f^{-1}(\alpha) \cap f^{-1}(\beta) = \varnothing$, so f really is a function
- Theorem: The ordinals defined by any CONS form an initial segment of the ordinals upto a limit ordinal
- Maximality Theorem: If (L', f') is a CONS, then there is a program T such that $T(L') \subseteq L$ and $f(T.x) = f'(x)$ for all x in L'

Recursiveness

- Let λ be the least ordinal not provided with a notation
- λ has to be a limit ordinal
- Can't we extend Kleene's system by adding a notation for λ?
- Say we use 5 to denote λ
- Then S.5 = e where {e} outputs a fundamental sequence for λ
- But then 3^e is already a notation for λ
- Question: how do we recognize if 3^e is a notation?
- Answer: this is an undecidable problem
- We can't even recognize if {e} is total, let alone a fundamental sequence
- Contrast this with ordinal notations in ACL2

Recursive Ordinal Notation Systems

- An ordinal is recursive if it is order-isomorphic to some woset $\langle W, < \rangle$ such that we can algorithmically determine for all a,b in W if $a<b$
- For example, show that ω^2 is recursive
- The set of recursive ordinals is countable
- The least non-recursive ordinal is a limit ordinal
- Recursive ordinals are a constructive analog of Cantor's well ordering approach to ordinals
- The constructive ordinals are a constructive analog of Cantor's ordinal generation principles
- Theorem: the recursive ordinals are exactly the constructive ordinals