Cardinals and Intro to Ordinals Notations

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Formal Methods, Lecture 5 September 2008

Last Time: Axiomatic Set Theory

- Axiom 0. Set Existence: $\langle \exists x :: x=x \rangle$
- Axiom 1. Extensionality: $\langle \forall x,y :: \langle \forall z :: z \in x \equiv z \in y \rangle \Rightarrow x=y \rangle$
- Axiom 3. Comprehension Scheme: For each formula φ without y free, the universal closure of the following is an axiom: $\langle \exists y :: \langle \forall x :: x \in y \equiv x \in z \wedge \varphi \rangle \rangle$
- Axiom 4. Pairing: $\langle \forall x,y :: \langle \exists z :: x \in z \wedge y \in z \rangle \rangle$
- Axiom 5. Union: $\langle \forall F :: \langle \exists A :: \langle \forall x,Y :: (Y \in F \wedge x \in Y) \Rightarrow x \in A \rangle \rangle \rangle$
- Axiom 6. Replacement Scheme. For each formula φ without Y free, the universal closure of the following is an axiom: $\langle \forall x \in A :: \langle \exists ! z :: \varphi(x,z) \rangle \rangle \Rightarrow \langle \exists Y :: \langle \forall x \in A :: \langle \exists z \in Y :: \varphi(x,z) \rangle \rangle \rangle$
- Axiom 7. Infinity: $\langle \exists x :: 0 \in x \wedge \langle \forall y \in x :: y+ \in x \rangle \rangle$ $(y+ = y \cup \{y\})$
- Axiom 9. Choice. $\langle \forall A :: \langle \exists R ::$ R well-orders A$\rangle \rangle$

Hey, What About Axiom 2?

- Axiom 2: Foundation. Sets are well-founded under $\in$
- $\langle \forall x :: \langle \exists y \in x \rangle \Rightarrow \langle \exists y \in x :: \neg \langle \exists z \in x :: z \in y \rangle \rangle \rangle$
- y is a minimal element above
- See handout for an alternate definition of foundation
- There are set theories where foundation is omitted
- So, we can have $x \in x$
- The problem is: when are two sets equal?
- What would you do?

Cardinals (& Axiom 8)

- $|A|$ is the least ordinal α s.t. $\alpha \approx A$ ($\approx$ means bijection)
- α is a cardinal iff $\alpha = |\alpha|$
- A cardinal number is an ordinal number whose size is > any of its elements
- A is finite iff $|A| < \omega$
- A is infinite iff it is not finite
- A is countable iff $|A| \leq \omega$
- A is uncountable iff it is not countable
- We can't prove that uncountable sets exist
- Axiom 8. Power Set. $\langle \forall x :: \langle \exists y :: \langle \forall z \subseteq x :: z \in y \rangle\rangle\rangle$
- $\wp(x) = \{z : z \subseteq x\}$

Properties of Cardinals

- Theorem: $|x| < |\wp(x)|$
- This allows us to obtain larger and larger cardinals
- Recall: a countable union of countable sets is countable
- If $|X_i| \leq \omega$ for all $i<\omega$, then $|\bigcup_{i<\omega} X_i| \leq \omega$
- If $\omega \leq \kappa$ and $|X_i| \leq \kappa$ for all $i<\kappa$, then $|\bigcup_{i<\kappa} X_i| \leq \kappa$
- $\kappa+$ is the least cardinal $>\kappa$ (overloaded notation!)
- Define ω_α (by transfinite recursion)
  - $\omega_0 = \omega$
  - $\omega_{\alpha+1} = (\omega_\alpha)+$ (successor cardinal)
  - $\omega_\gamma = \cup\{\omega_\alpha : \alpha < \gamma\}$ (limit cardinal)
- Note $\omega_1 >>> \varepsilon_0$

Extended Initial Sequence of Ordinals

- $0, 1, 2, ..., \omega,$
- $\omega+1, \omega+2, ..., \omega+\omega = \omega\cdot 2, \omega\cdot 2+1, ..., \omega\cdot 3, ...,$
- $\omega\cdot\omega = \omega^2, ..., \omega^3, ..., \omega^\omega, ...,$
- $\varepsilon_0, \varepsilon_1, \varepsilon_2, ..., \varepsilon_\omega, ..., \varepsilon_{\varepsilon 0}, ..., ..., ...,$ (big jumps)
- $\omega_1, \omega_2, \omega_3, ..., \omega_\omega, ...,$ (huge jumps)
- $\omega_{\varepsilon 0}, ..., \omega_\sigma = \sigma$
- Wait, does such a thing as σ exist?
- Does $f(\alpha) = \omega_\alpha$ have a fixpoint?
- This question highlights why fundamental questions about termination are really questions about ordinals

Limits

- Let λ be a limit ordinal and let $\langle \alpha_\xi : \xi < \lambda \rangle$ be an increasing sequence of ordinals. Then the sequence has a limit: $\lim \langle \alpha_\xi : \xi < \lambda \rangle = \cup\{\alpha_\xi : \xi < \lambda\}$
- Let f: $\lambda \rightarrow \lambda$, where α is a limit ordinal in λ. We say f is continuous at α iff $f(\alpha) = \cup\{f(\xi) : \xi < \alpha\}$
- A function f: $\lambda \rightarrow \lambda$ is normal iff it is order preserving and continuous at every limit ordinal in λ
- Let f: On $\rightarrow$ On be a normal function. For every α there is a fixed-point γ of f such that $\gamma \geq \alpha$
- $\omega_0, \omega_1, \omega_2, \omega_3, \ldots, \omega_\omega, \ldots, \omega_{\varepsilon 0}, \ldots,$ has a fixed point $\omega_\sigma = \sigma$
- Are there ordinals λ so that the following holds?
- Lef f: $\lambda \rightarrow \lambda$ be a normal function. For every $\alpha<\lambda$ there is a fixed-point γ of f such that $\gamma \geq \alpha$

Cofinality

- $X \subseteq Y$ is *cofinal* in Y iff for all y in Y there is x in X s.t. $y \leq x$
- $f: \alpha \to \beta$ maps α to β cofinally iff ran(f) is cofinal in β
- The cofinality of β, $cf(\beta)$, is the least α such that there is a map from α cofinally into β
- Note: $cf(\beta) \leq \beta$; If β is a successor, $cf(\beta) = 1$
- β is regular iff β is a limit ordinal and $cf(\beta) = \beta$
- Lemma: If β is regular, β is a cardinal
- ω, κ+ are regular
- So: $|A| < \omega_\gamma$ and $\alpha \in A \Rightarrow \alpha < \omega_\gamma$ implies $\lim A < \omega_\gamma$
- So: ω_1 is an ordinal that satisfies our previous question
- Are there regular limit cardinals (weakly inaccessibles)?

Veblen Hierarchy

- Consider $\varphi_0(\alpha) = \omega^\alpha$
- Which gives rise to 1, ω^1, ω^2, ω^3, ..., ω^ω, ..., $\omega^{\varepsilon 0}$
- Define $\varphi_1(\alpha)$ so that it enumerates the fixpoints of φ_0
- Which gives rise to ε_0, ε_1, ε_2, ..., ε_ω, ...
- $\varphi_2(\alpha)$ enumerates the fixpoints of φ_1, ...
- $\varphi_{\gamma+1}(\alpha)$ enumerates the fixpoints of φ_γ
- $\varphi_\beta(\alpha)$ enumerates common fixpoints of φ_γ for $\gamma < \beta$
- $\varphi_\alpha(\beta) < \varphi_\gamma(\delta)$ iff
  - $\alpha=\gamma$ and $\beta < \delta$, or
  - $\alpha<\gamma$ and $\beta < \varphi_\gamma(\delta)$, or
  - $\alpha>\gamma$ and $\varphi_\alpha(\beta) < \delta$

The Ordinal Γ_0

- Γ enumerates ordinals s.t. $\varphi_\alpha(0) = \alpha$
- Γ_0 is the smallest such ordinal
- Γ_0 plays a key role in proof theory

Ordinal Notations

- An ordinal notation for ordinal α is an explicit, constructive bijection between ω and α
- It turns out that there are ordinal notations for the Veblen hierarchy of countable ordinals
- Project:
  - Extend ACL2 with notations up to at least Γ_0
  - Define ordinal arithmetic for these notations
- What does ACL2 currently do?
- It only has ordinals up to ε_0

Cantor Normal Form

- Theorem (Cantor Normal Form)

  For every ordinal $\alpha \in \varepsilon_0$, there are unique $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$, $p \in \omega$, and $x_1, ..., x_n \in \omega \setminus \{0\}$ such that:
  - $\alpha > \alpha_1$
  - $\alpha = \omega^{\alpha 1} x_1 + \cdots + \omega^{\alpha n} x_n + p$
- This is the (finite) representation of ε_0 in ACL2
- Lexicographic ordering can be embedded in ω^ω
  - $\langle x,y,z \rangle$ becomes $\omega^2 x + \omega y + z$
- ACL2 also includes algorithms for ordinal arithmetic on ordinal notations
- These algorithms are quite interesting
- ACL2 can also reason about the ordinals upto ε_0
- Try it out