

Lecture 23

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Unification for FOL

- ▶ Let C be a clause; if we negate all literals in C , we get C^-
- ▶ A unifier for a clause $C=\{l_1,\dots,l_n\}$ is a unifier for $\{(l_1,l_2), (l_2, l_3), \dots, (l_{n-1},l_n)\}$
- ▶ Let C, D be clauses (assume there are no common variables since we can rename vars). K is a **U-resolvent** of C, D iff there are non-empty $\underline{C}'\subseteq C, \underline{D}'\subseteq D$ s.t. σ is a unifier for $\underline{C}'\cup\underline{D}'^-$ and $K=(C\setminus\underline{C}' \cup D\setminus\underline{D}')\sigma$. Note $|\underline{C}'|, |\underline{D}'|$ can be >1

$$C = \{ \neg R(x), R(f(x)) \} \quad D = \{ \neg R(f(f(x))), P(x) \} \quad \text{corresponds to}$$

$$\langle \forall x (\neg R(x) \vee R(f(x))) \wedge (\neg R(f(f(x))) \vee P(x)) \rangle \quad \text{equivalent to}$$

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so I will rename variables in clauses as I see fit

Recall from the Prenex Normal Form algorithm (let z,y be x in the example)

$$\langle \forall x :: \phi \rangle \wedge \langle \forall y :: \psi \rangle \equiv \langle \forall z :: \phi \frac{z}{x} \wedge \psi \frac{z}{y} \rangle \quad \text{where } z \text{ is not free in LHS}$$

U-resolvent example

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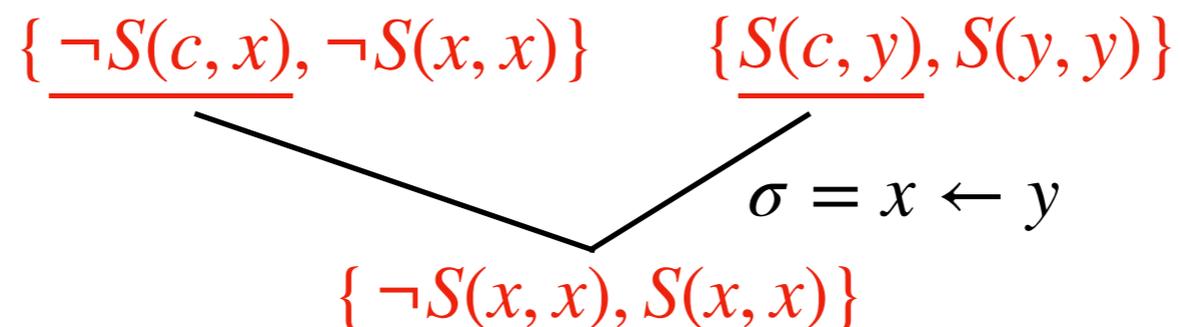
$$\{ \neg R(x), \underline{R(f(x))} \} \quad \{ \underline{\neg R(f(f(y)))}, P(y) \}$$

$$\begin{array}{c} \swarrow \quad \searrow \\ \sigma = f(y) \leftarrow x \\ \{ \neg R(f(y)), P(y) \} \end{array}$$

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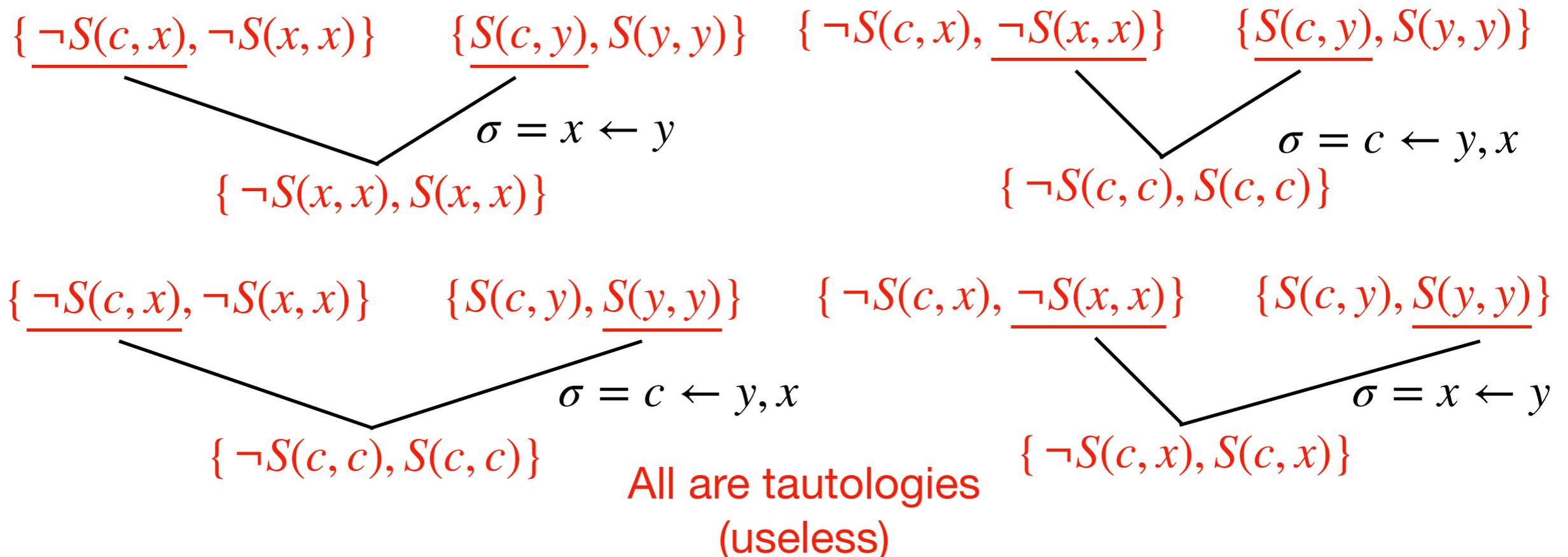
One possible U-resolution step



Tautology, so useless

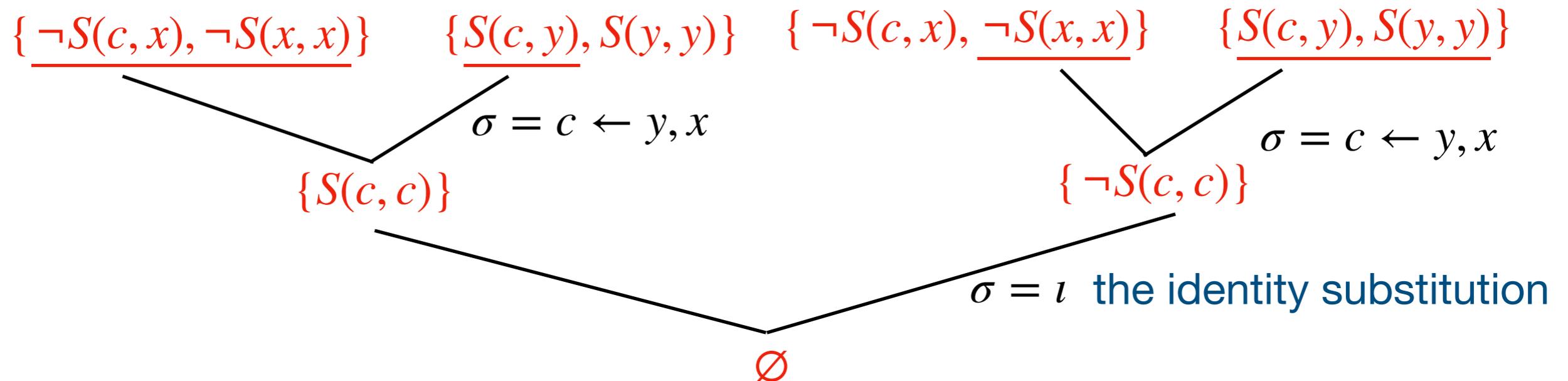
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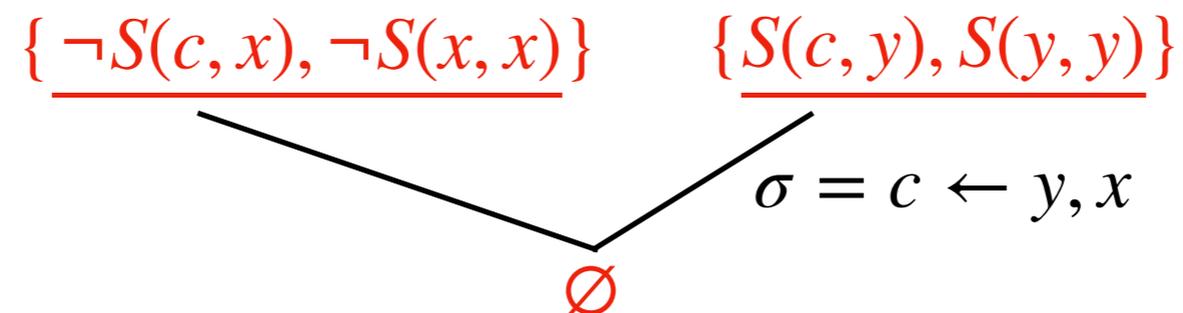
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- ▶ This is the Barber of Seville problem: Prove that there is no barber who shaves all those, and those only, who do not shave themselves.

$$\neg \langle \exists b \langle \forall x S(b, x) \equiv \neg S(x, x) \rangle \rangle$$

Unification for FOL

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- ▶ Lemma: Let C, D be clauses. Then
 - ▶ every resolvent of ground instances of C, D is a ground instance of a U-resolvent of C, D
 - ▶ every ground instance of a U-resolvent of C, D is a resolvent of ground instances of C, D
- ▶ Let \mathcal{K} be a set of ground clauses, $\text{Res}(\mathcal{K})=\mathcal{K} \cup \{K \mid K \text{ is a resolvent of } C,D\in\mathcal{K}\}$
- ▶ Let \mathcal{K} be a set of FO clauses, $\text{URes}(\mathcal{K})=\mathcal{K} \cup \{K \mid K \text{ is a U-resolvent of } C,D\in\mathcal{K}\}$
- ▶ Let $\text{URes}_0(\mathcal{K})=\mathcal{K}$, $\text{URes}_{n+1}(\mathcal{K})=\text{URes}(\text{URes}_n(\mathcal{K}))$, $\text{URes}_\omega(\mathcal{K})=\bigcup_{n\in\omega}\text{URes}_n(\mathcal{K})$

Unification for FOL

- ▶ Let C, D be clauses (assume there are no common variables since we can rename vars). K is a **U-resolvent** of C, D iff there are non-empty $\underline{C}' \subseteq C, \underline{D}' \subseteq D$ s.t. σ is a unifier for $\underline{C}' \cup \underline{D}'$ and $K = (C \setminus \underline{C}' \cup D \setminus \underline{D}')\sigma$. Note $|\underline{C}'|, |\underline{D}'|$ can be >1
- ▶ $G(K)$ is the set of ground instances of K , $G(\mathcal{K}) = \bigcup_{K \in \mathcal{K}} G(K)$
- ▶ Lemma: $\text{Res}_n(G(\mathcal{K})) = G(\text{URes}_n(\mathcal{K}))$ and $\text{Res}_\omega(G(\mathcal{K})) = G(\text{URes}_\omega(\mathcal{K}))$
- ▶ Lemma: $\emptyset \in \text{Res}_\omega(G(\mathcal{K}))$ iff $\emptyset \in \text{URes}_\omega(\mathcal{K})$
- ▶ For Φ a set of \forall formulas in CNF: $G(\mathcal{K}(\Phi)) = \mathcal{K}(G(\Phi))$, where $\mathcal{K}(\Phi)$ is set-representation of CNF
- ▶ Theorem: For Φ a set of \forall formulas in CNF, Φ is Sat iff $\emptyset \notin \text{URes}_\omega(\mathcal{K}(\Phi))$
 - ▶ Proof: Φ is Sat iff $G(\Phi)$ is (propositionally) Sat iff $\mathcal{K}(G(\Phi))$ is Sat iff $G(\mathcal{K}(\Phi))$ is Sat iff $\emptyset \notin \text{Res}_\omega G(\mathcal{K}(\Phi))$ iff $\emptyset \notin \text{URes}_\omega \mathcal{K}(\Phi)$

FOL Checking with Unification

- ▶ FO validity checker: Given FO ϕ , negate & Skolemize to get universal ψ s.t. $\text{Valid}(\phi)$ iff $\text{UNSAT}(\psi)$. Let G be the set of ground instances of ψ (possibly infinite, but countable). Let $G_1, G_2 \dots$, be a sequence of finite subsets of G s.t. $\forall g \subseteq G, |g| < \omega, \exists n$ s.t. $g \subseteq G_n$. $\exists n$ s.t. $\text{Unsat } G_n$ iff $\text{Unsat } \psi$ (and $\text{Valid } \phi$)
- ▶ Unification: intelligently instantiate formulas
- ▶ FO validity checker w/ unification: Given FO ϕ , negate & Skolemize to get universal ψ s.t. $\text{Valid}(\phi)$ iff $\text{UNSAT}(\psi)$. **Convert ψ into equivalent CNF \mathcal{K} .**
Then, $\text{Unsat } \psi$ iff $\emptyset \in \text{URes}_\omega(\mathcal{K})$ iff $\exists n$ s.t. $\emptyset \in \text{URes}_n(\mathcal{K})$.
- ▶ We say that U-resolution is *refutation-competete*: If $\text{Unsat}(\mathcal{K})$ then there is a proof using U-resolution (*i.e.*, you can derive \emptyset), so we have a semi-decision procedure for validity.

FOL Checking Examples

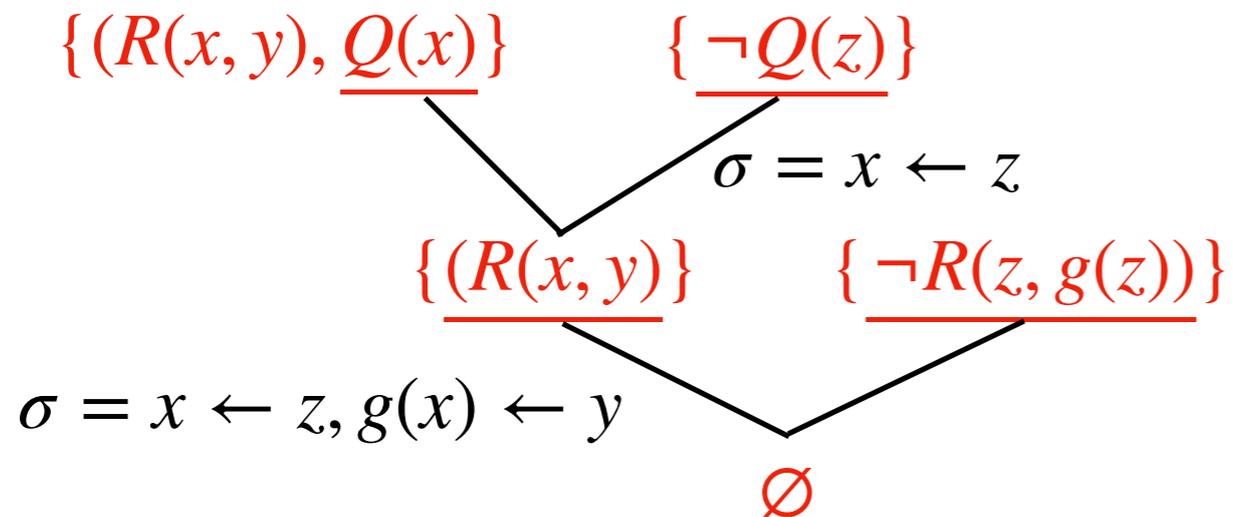
- FO validity checker w/ unification: Given FO ϕ , negate & Skolemize to get universal ψ s.t. $\text{Valid}(\phi)$ iff $\text{UNSAT}(\psi)$. Convert ψ into equivalent CNF \mathcal{K} .

Then, $\text{Unsat}(\psi)$ iff $\emptyset \in \text{URes}_\omega(\mathcal{K})$ iff $\exists n$ s.t. $\emptyset \in \text{URes}_n(\mathcal{K})$.

$$\phi = \neg \langle \forall x, y (R(x, y) \vee Q(x)) \wedge \neg R(x, g(x)) \wedge \neg Q(y) \rangle$$

$$\psi = \langle \forall x, y (R(x, y) \vee Q(x)) \wedge \neg R(x, g(x)) \wedge \neg Q(y) \rangle$$

$$\mathcal{K} = \{ \{R(x, y), Q(x)\}, \{ \neg R(x, g(x)) \}, \{ \neg Q(y) \} \}$$



Let C, D be clauses (w/ no common variables). K is a U-resolvent of C, D iff there are non-empty $\underline{C'} \subseteq C, \underline{D'} \subseteq D$ s.t. σ is a unifier for $\underline{C'} \cup \underline{D'}$ and $K = (C \setminus \underline{C'} \cup D \setminus \underline{D'})\sigma$.

Recall

So, $\text{Unsat}(\psi)$ and $\text{Valid}(\phi)$