Lecture 23

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Unification for FOL

- Let C be a clause; if we negate all literals in C, we get C^-
- A unifier for a clause $C=\{l_1,\ldots,l_n\}$ is a unifier for $\{(l_1,l_2), (l_2, l_3), \ldots, (l_{n-1},l_n)\}$
- Let C, D be clauses (assume there are no common variables since we can rename vars). K is a U-resolvent of C, D iff there are non-empty $\underline{C}' \subseteq C$, $\underline{D}' \subseteq D$ s.t. σ is a unifier for $\underline{C}' \cup \underline{D}'^-$ and $K=(C \backslash \underline{C}' \cup D \backslash \underline{D}')\sigma$. Note $|\underline{C}'|$, $|\underline{D}'|$ can be >1

$$C = \{\neg R(x), R(f(x))\} \;\; D = \{\neg R(f(f(x))), P(x)\}$$ corresponds to

$$\langle \forall x\ (\neg R(x) \vee R(f(x))) \;\wedge\; (\neg R(f(f(x))) \vee P(x))\rangle$$ equivalent to

$$\langle \forall x\ \neg R(x) \vee R(f(x))\rangle \;\wedge\; \langle \forall x\ \neg R(f(f(x))) \vee P(x)\rangle$$ equivalent to

$$\langle \forall x\ \neg R(x) \vee R(f(x))\rangle \;\wedge\; \langle \forall y\ \neg R(f(f(y))) \vee P(y)\rangle$$ corresponds to

$$C = \{\neg R(x), R(f(x))\} \;\; D = \{\neg R(f(f(y)) \vee P(y)\}$$

so I will rename variables in clauses as I see fit

Recall from the Prenex Normal Form algorithm (let z,y be x in the example)

$$\langle \forall x :: \phi\rangle \wedge \langle \forall y :: \psi\rangle \;\equiv\; \langle \forall z :: \phi\frac{z}{x} \wedge \psi\frac{z}{y}\rangle$$ **where z is not free in LHS**

U-resolvent example

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$$C = \{\neg R(x), R(f(x))\} \quad D = \{\neg R(f(f(x))), P(x)\}$$

$$\{\neg R(x), \underline{R(f(x))}\} \qquad \{\underline{\neg R(f(f(y)))}, P(y)\}$$

$$\sigma = f(y) \leftarrow x$$

$$\{\neg R(f(y)), P(y)\}$$

U-resolvent example

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- Try this: $C = \{\neg S(c,x), \neg S(x,x)\},\ D = \{S(x,x), S(c,x)\}$

One possible U-resolution step

$$\{\underline{\neg S(c,x)}, \neg S(x,x)\} \qquad \{\underline{S(c,y)}, S(y,y)\}$$

$$\sigma = x \leftarrow y$$

$$\{\neg S(x,x), S(x,x)\}$$

Tautology, so useless

U-resolvent example

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$\{\underline{\neg S(c,x)}, \neg S(x,x)\}$ $\quad$ $\{\underline{S(c,y)}, S(y,y)\}$ $\quad \sigma = x \leftarrow y$

$\{\neg S(x,x), S(x,x)\}$

$\{\neg S(c,x), \underline{\neg S(x,x)}\}$ $\quad$ $\{\underline{S(c,y)}, S(y,y)\}$ $\quad \sigma = c \leftarrow y, x$

$\{\neg S(c,c), S(c,c)\}$

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$\{\neg S(c,x), S(c,x)\}$

All are tautologies
(useless)

U-resolvent example

- Let *C* be a clause; if we negate all literals in *C*, we get C^-
- A unifier for a clause $C=\{l_1,\ldots,l_n\}$ is a unifier for $\{(l_1,l_2), (l_2, l_3), \ldots, (l_{n-1},l_n)\}$
- Let *C, D* be clauses (assume there are no common variables since we can rename vars). *K* is a U-resolvent of *C*, *D* iff there are non-empty $\underline{C'}\subseteq C$, $\underline{D'}\subseteq D$ s.t. σ is a unifier for $\underline{C'}\cup\underline{D'}^-$ and $K=(C\backslash\underline{C'} \cup D\backslash\underline{D'})\sigma$. Note $|\underline{C'}|$, $|\underline{D'}|$ can be >1
- Try this: $C = \{\neg S(c,x), \neg S(x,x)\},\ D = \{S(x,x), S(c,x)\}$

$\{\underline{\neg S(c,x), \neg S(x,x)}\}$ $\{\underline{S(c,y)}, S(y,y)\}$ $\{\neg S(c,x), \underline{\neg S(x,x)}\}$ $\{\underline{S(c,y), S(y,y)}\}$

$\sigma = c \leftarrow y, x$ $\sigma = c \leftarrow y, x$

$\{S(c,c)\}$ $\{\neg S(c,c)\}$

$\sigma = \iota$ the identity substitution

$\emptyset$

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- Try this: $C = \{\neg S(c,x), \neg S(x,x)\},\ D = \{S(x,x), S(c,x)\}$

$$\{\underline{\neg S(c,x), \neg S(x,x)}\} \qquad \{\underline{S(c,y), S(y,y)}\}$$

$$\sigma = c \leftarrow y, x$$

$$\emptyset$$

- This is the Barber of Seville problem: Prove that there is no barber who shaves all those, and those only, who do not shave themselves.

$$\neg\langle\, \exists b\ \langle \forall x\ S(b,x) \equiv \neg S(x,x)\rangle\rangle$$

Unification for FOL

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- Lemma: Let C, D be clauses. Then
  - every resolvent of ground instances of C, D is a ground instance of a U-resolvent of C, D
  - every ground instance of a U-resolvent of C, D is a resolvent of ground instances of C, D
- Let $\mathcal{K}$ be a set of ground clauses, $\text{Res}(\mathcal{K})=\mathcal{K} \cup \{K \mid K \text{ is a resolvent of } C,D \in \mathcal{K}\}$
- Let $\mathcal{K}$ be a set of FO clauses, $\text{URes}(\mathcal{K})=\mathcal{K} \cup \{K \mid K \text{ is a U-resolvent of } C,D \in \mathcal{K}\}$
- Let $\text{URes}_0(\mathcal{K})=\mathcal{K}$, $\text{URes}_{n+1}(\mathcal{K})=\text{URes}(\text{URes}_n(\mathcal{K}))$, $\text{URes}_\omega(\mathcal{K})=\cup_{n\in\omega}\text{URes}_n(\mathcal{K})$

Unification for FOL

- Let C, D be clauses (assume there are no common variables since we can rename vars). K is a U-resolvent of C, D iff there are non-empty $\underline{C}' \subseteq C$, $\underline{D}' \subseteq D$ s.t. σ is a unifier for $\underline{C}' \cup \underline{D}'^{-}$ and $K = (C \setminus \underline{C}' \cup D \setminus \underline{D}')\sigma$. Note $|\underline{C}'|$, $|\underline{D}'|$ can be >1
- $G(K)$ is the set of ground instances of K, $G(\mathscr{K}) = \cup_{K \in \mathscr{K}} G(K)$
- Lemma: $\mathrm{Res}_n(G(\mathscr{K})) = G(\mathrm{URes}_n(\mathscr{K}))$ and $\mathrm{Res}_\omega(G(\mathscr{K})) = G(\mathrm{URes}_\omega(\mathscr{K}))$
- Lemma: $\varnothing \in \mathrm{Res}_\omega(G(\mathscr{K}))$ iff $\varnothing \in \mathrm{URes}_\omega(\mathscr{K})$
- For Φ a set of ∀ formulas in CNF: $G(\mathscr{K}(\Phi)) = \mathscr{K}(G(\Phi))$, where $\mathscr{K}(\Phi)$ is set-representation of CNF
- Theorem: For Φ a set of ∀ formulas in CNF, Φ is Sat iff $\varnothing \notin \mathrm{URes}_\omega(\mathscr{K}(\Phi))$
  - Proof: Φ is Sat iff G(Φ) is (propositionally) Sat iff $\mathscr{K}(G(\Phi))$ is Sat iff $G(\mathscr{K}(\Phi))$ is Sat iff $\varnothing \notin \mathrm{Res}_\omega G(\mathscr{K}(\Phi))$ iff $\varnothing \notin \mathrm{URes}_\omega \mathscr{K}(\Phi)$

FOL Checking with Unification

- FO validity checker: Given FO ϕ, negate & Skolemize to get universal ψ s.t. Valid(ϕ) iff UNSAT(ψ). Let G be the set of ground instances of ψ (possibly infinite, but countable). Let G_1, G_2 …, be a sequence of finite subsets of G s.t. $\forall g \subseteq G, |g| < \omega$, $\exists n$ s.t. $g \subseteq G_n$. $\exists n$ s.t. Unsat G_n iff Unsat ψ (and Valid ϕ)
- Unification: intelligently instantiate formulas
- FO validity checker w/ unification: Given FO ϕ, negate & Skolemize to get universal ψ s.t. Valid(ϕ) iff UNSAT(ψ). Convert ψ into equivalent CNF $\mathscr{K}$. Then, Unsat ψ iff $\emptyset \in URes_\omega(\mathscr{K})$ iff $\exists n$ s.t. $\emptyset \in URes_n(\mathscr{K})$.
- We say that U-resolution is *refutation-compete*: If Unsat($\mathscr{K}$) then there is a proof using U-resolution (*i.e.*, you can derive ∅), so we have a semi-decision procedure for validity.

FOL Checking Examples

- FO validity checker w/ unification: Given FO ϕ, negate & Skolemize to get universal ψ s.t. Valid(ϕ) iff UNSAT(ψ). Convert ψ into equivalent CNF $\mathscr{K}$. Then, Unsat(ψ) iff $\emptyset \in \text{URes}_\omega(\mathscr{K})$ iff $\exists n$ s.t. $\emptyset \in \text{URes}_n(\mathscr{K})$.

$$\phi = \neg\langle \forall x, y\ (R(x,y) \vee Q(x)) \ \wedge\ \neg R(x, g(x)) \ \wedge\ \neg Q(y)\rangle$$

$$\psi = \langle \forall x, y\ (R(x,y) \vee Q(x)) \ \wedge\ \neg R(x, g(x)) \ \wedge\ \neg Q(y)\rangle$$

$$\mathscr{K} = \{\{R(x,y), Q(x)\}, \{\neg R(x, g(x))\}, \{\neg Q(y)\}\}$$

$\{(R(x,y), \underline{Q(x)}\}$ $\{\underline{\neg Q(z)}\}$

$\sigma = x \leftarrow z$

$\{\underline{(R(x,y)}\}$ $\{\underline{\neg R(z, g(z))}\}$

$\sigma = x \leftarrow z, g(x) \leftarrow y$

$\emptyset$

So, Unsat(ψ) and Valid(ϕ)

Let *C*, *D* be clauses (w/ no common variables). *K* is a U-resolvent of *C*, *D* iff there are non-empty $\underline{C'} \subseteq C$, $\underline{D'} \subseteq D$ s.t. σ is a unifier for $\underline{C'} \cup \underline{D'}^-$ and $K = (C \backslash \underline{C'} \cup D \backslash \underline{D'})\sigma$.

Recall