Interpretations

Definition 1 *An* S-interpretation $\mathcal{J}$ *is a pair* $\langle \mathbf{U}, \beta \rangle$, *where* $\mathbf{U} = \langle A, \mathbf{a} \rangle$ *is an* S*-structure and* $\beta : \mathit{Var} \rightarrow A$, *is an* assignment, *a function that assigns values to the variables.*

We define the meaning of any term t in interpretation $\mathcal{J}$, denoted $\mathcal{J}.t$, as follows.

1. If $v \in \mathit{Var}$, then $\mathcal{J}.v = \beta.v$.
2. If $c \in S$ is a constant symbol, then $\mathcal{J}.c = c^{\mathbf{U}}$.
3. If $ft_1 \ldots t_n$ is a term, then $\mathcal{J}(ft_1 \ldots t_n)$ is $(f^{\mathbf{U}})(\mathcal{J}.t_1, \ldots, \mathcal{J}.t_n)$.

If β is an assignment, then $\beta\frac{a}{x}(y)$ is a if $y = x$ and $\beta.y$ otherwise. For $\mathcal{J} = \langle \mathbf{U}, \beta \rangle$, $\mathcal{J}\frac{a}{x}$ denotes $\langle \mathbf{U}, \beta\frac{a}{x} \rangle$.

Satisfaction

1. $\mathcal{J} \models (t_1 \equiv t_2)$ iff $\mathcal{J}.t_1 \;=\; \mathcal{J}.t_2$.
2. $\mathcal{J} \models R(t_1 \ldots t_n)$ iff $\langle \mathcal{J}.t_1, \ldots, \mathcal{J}.t_n \rangle \in R^{\mathbf{U}}$.
3. $\mathcal{J} \models \neg\varphi$ iff not $\mathcal{J} \models \varphi$.
4. $\mathcal{J} \models (\varphi \vee \psi)$ iff $\mathcal{J} \models \varphi$ or $\mathcal{J} \models \psi$.
5. $\mathcal{J} \models (\varphi \wedge \psi)$ iff $\mathcal{J} \models \varphi$ and $\mathcal{J} \models \psi$.
6. $\mathcal{J} \models (\varphi \rightarrow \psi)$ iff if $\mathcal{J} \models \varphi$ then $\mathcal{J} \models \psi$.
7. $\mathcal{J} \models (\varphi \leftrightarrow \psi)$ iff $\mathcal{J} \models \varphi$ if and only if $\mathcal{J} \models \psi$.
8. $\mathcal{J} \models \exists x\varphi$ iff for some $a \in A$, $\mathcal{J}\frac{a}{x} \models \varphi$.
9. $\mathcal{J} \models \forall x\varphi$ iff for any $a \in A$, $\mathcal{J}\frac{a}{x} \models \varphi$.

If $\mathcal{J} \models \varphi$ we say that φ holds in $\mathcal{J}$; we also say that $\mathcal{J}$ is a model of φ; we also say that $\mathcal{J}$ satisfies φ .

Given, Φ, a set of formulas, $\mathcal{J} \models \Phi$ ($\mathcal{J}$ is a model of Φ) iff for every $\varphi \in \Phi$, $\mathcal{J} \models \varphi$.

You should convince yourself that $\mathcal{J} \models \varphi$ iff φ is *true* under interpretation $\mathcal{J}$.

Note that the meaning of a sentence does not depend on the assignment. In general, we are interested in sentences, but to evaluate them, we have to evaluate subformulas, which may not be sentences, therefore, the need for assignments. This kind of thing comes up in programming a lot.

Consequence

Definition 2 *Let Φ be a set of formulas and φ a formula. Then $\Phi \models \varphi$ (φ is a consequence of Φ) iff for every interpretation, $\mathcal{J}$, which is a model of Φ, we have that $\mathcal{J} \models \varphi$.*

We have developed enough mathematical machinery to reconsider, in a more rigourous way, one of our initial goals.

Recall, that we were interested in whether $\Phi \models \varphi$ iff $\Phi \vdash \varphi$. For example, we saw a proof that groups have a left inverse, and you should be convinced that such a proof implies $\Phi_{gr} \models \forall v_o \exists v_1 (v_1 \circ v_0) \equiv e$, where $\Phi_{gr} = \{\forall v_0 \forall v_1 \forall v_2 (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2), \forall v_0 v_0 \circ e \equiv v_0, \forall v_0 \exists v_1 v_0 \circ v_1 = e\}$. Once we develop the notion of proof, this will be an easy theorem.

What is not as clear is whether the opposite direction holds. The completeness theorem will establish this. That comes after we define what a proof is and will be the first main theorem we prove.

Definition 3 *A formula φ is* valid *iff $\emptyset \models \varphi$, which we abbreviate by $\models \varphi$.*

Definition 4 *A formula φ is* satisfiable, *written Sat φ iff there is an interpretation which is a model of φ; similarly, a set of formulas Φ is satisfiable, Sat Φ iff there is an interpretation which is a model of all the formulas in Φ.*

Lemma 1 *For all Φ and all φ, $\Phi \models \varphi$ iff not Sat $\Phi \cup \{\neg\varphi\}$.*

As a consequence, φ is valid iff $\neg\varphi$ is not satisfiable.

Definition 5 *Formulas φ and ψ are* logically equivalent, *written $\varphi \dashv\models \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$.*

Note that this is equivalent to saying that φ and ψ are logically equivalent iff $\models \varphi \leftrightarrow \psi$.

It should be clear that the only boolean connectives we need are $\neg$ and $\vee$. Why?

In addition, we do not need $\forall$. Why?

Coincidence Lemma

Lemma 2 *(Coincidence Lemma). Let* $\mathcal{I}_1 = \langle \mathbf{U}_1, \beta_1 \rangle$ *be an* S_1*-interpretation and and* $\mathcal{I}_2 = \langle \mathbf{U}_2, \beta_2 \rangle$ *be an* S_2*-interpretation, both with the same domain. Let* $S = S_1 \cap S_2$.

1. *Let* t *be an* S*-term. If* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *agree on the* S*-symbols occurring in* t *and on the variables occuring in* t*, then* $\mathcal{I}_1(t) = \mathcal{I}_2(t)$.

2. *Let* φ *be an* S*-formula. If* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *agree on the* S*-symbols and on the variables occurring free in* φ*, then* $\mathcal{I}_1 \models \varphi$ *iff* $\mathcal{I}_2 \models \varphi$.

Proof By induction on S-terms and then on S-formulas. $\square$