Public Key Cryptosystems Based on Discrete Logarithm Problem

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Textbook: "Cryptography: Theory and Applications",
Douglas Stinson, Chapman & Hall/CRC Press, 2002

Reading: Chapter 6, Sections 6.1-6.4, and 6.7.3

Outline

- El Gamal Cryptosystem
- Algorithms for Discrete Logarithm
- Implementation Issues
- Diffie-Hellman Problems and Key Establishment

Element for El Gamal Scheme

- Motivation of design
  - RSA is based on the difficulty of factoring large numbers
  - El Gamal scheme is based on the difficulty of computing discrete logarithms
- Order of an element of a multiplicative group (G, .):
  - $<\alpha> = \{\alpha^i : 0 \le i \le n-1\}$; n is the order of α
- Discrete Logarithm:
  - Given a multiplicative group $(G, .)$, an element $\alpha \in G$ with order n, and an element $\beta \in G$ s.t. $\alpha^a = \beta$
  - Question: find the unique integer $0 \le a \le n-1$ s.t. $\alpha^a = \beta$
    - This is the same as finding $\log_\alpha(\beta)$

El Gamal Cryptosystem

- Cryptosystem
  - p prime s.t. $(Z_p^*, .)$ discrete logarithm is infeasible
  - Let α be a primitive element
  - $\mathcal{P} = Z_p^*$; $\mathcal{C} = Z_p^* \times Z_p^*$;
  - $\mathcal{K} = \{(p, \alpha, a, \beta) : \beta \equiv \alpha^a \bmod p\}$
  - Public: p, α, β; Private: a;
  - For $K = (p, \alpha, a, \beta)$ and a secret number $k \in Z_p^*$;
  - $e_K(x, k) = (y_1, y_2)$ s.t.
    - $y_1 = \alpha^k \bmod p$ and $y_2 = x\beta^k \bmod p$;
  - $d_K(y_1, y_2) = ?$

Example:

- $p = 2579$;
- $\alpha = 2$ (primitive element modulo p)
- $a = 765$
- $\beta = 2^{765} \bmod 2579 = 949$
- Encrypt $x = 1299$; $k = 853$
  - $y_1 = 2^{853} \bmod 2579 = 435$; $y_2 = 1299 * 949^{853} \bmod 2579 = 2396$
- Decrypt $(y_1, y_2) = (435, 2396)$
  - $x = 2396/435^{765} \bmod 2579 = 1299$

Algorithms for Discrete Logarithm

- El Gamal cryptosystem would be insecure if we can compute the discrete logarithm
- Discrete logarithm is believed to be infeasible if:
  - p is carefully chosen against known attacks
  - α is a primitive element modulo p
  - Example:
    - p has at least 300 digits,
    - p-1 has at least one "large" prime factor

Algorithms for Discrete Logarithm

- Assumption:
  - Multiplication in G can be done in $O(1)$
- Exhaustive search: Cost = $O(n)$
- Shank's Algorithm (G, n, α, β) [time-memory tradeoff]
  - $m \leftarrow \lceil \sqrt{n} \rceil$
  - **For** j=0 **to** m-1 **do** Compute α^{mj}
  - **Sort** the m pairs (j, α^{mj}) with respect to second coordinate $\Rightarrow$ List L_1
  - **For** i=0 **to** m-1 **do** compute $\beta\alpha^{-i}$
  - **Sort** the m pairs $(j, \beta\alpha^{-i})$ with respect to second coordinate $\Rightarrow$ List L_2
  - **Find** a pair $(j, y) \in L_1$ and a pair $(i, y) \in L_2$ [Note: same y]
  - $\text{Log}_\alpha\beta = (mj+i) \bmod n$
- Complexity of Shank's algorithm: Time? Space?

Algorithms for Discrete Logarithm

- Pollard Rho Discrete log
  - Time: $O(\sqrt{n})$
- Pohlig-Hellman Algorithm
  - Time: $O(\max(c_i\sqrt{q_i}))$ s.t. $n = q_1^{c1} \ldots q_k^{ck}$
- Index Calculus Method:
  - Specialized algorithm for Z_p^* and primitive element α
  - Idea:
    - Use a factor base $B = \{p_1, p_2, \ldots, p_B\}$
    - Find the logarithms of the primes in the factor base
    - Use these logarithms to compute the logarithm of β
- Lower bound on generic algorithms:
  - Definition: a generic algorithm applies to any group and does not use any properties of the element of the group such as factorization, …
  - Any generic algorithm for discrete logarithm has a lower bound on time complexity: $\Omega(\sqrt{n})$

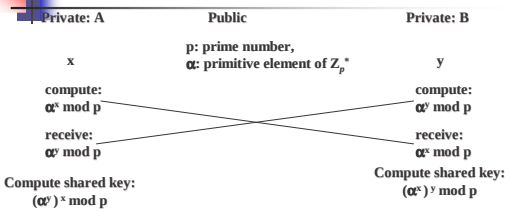
Discrete Logarithm Algorithms in Practice

- Setups:
  - $G = (Z_p^*, .)$, p prime, α primitive element modulo p
  - $G = (Z_p^*, .)$, p and q prime ($p \equiv 1 \bmod q$), α element having order p
  - $G = (F_{2^r}^*, .)$, α primitive element modulo in $F_{2^r}^*$
  - Elliptic Curves modulo a prime or over a finite field
- Lenstra and Verheul report to be secure until year 2020:
  - $p = 2^{160}$ for elliptic curves
  - $p = 2^{1880}$ for $(Z_p^*, .)$
  - Elliptic Curve implementations are the most efficient
    - Mainly due to inexistence of an index calculus attack
    - Adequate for low power/resources devices such as PDAs and smartcards
- Latest challenge:
  - ECC2K-108 over $F_{2^{109}}^*$ (solved in April 2000) using 9500 computers about 50 times the computation effort required to factor the RSA challenge RSA-512

Diffie-Hellman Problems

- Computational Diffie-Hellman
  - Given a multiplicative group $(G, .)$, an element $\alpha \in G$ (order n), two elements $\alpha^b, \alpha^c \in <\alpha>$
  - Question: find α^{bc}
- Decisional Diffie-Hellman
  - Given a multiplicative group $(G, .)$, an element $\alpha \in G$ (order n), three elements $\alpha^b, \alpha^c, \alpha^d \in <\alpha>$
  - Question: Is $d = bc$?
- Turing Reductions:
  - Decision Diffie-Hellman can be reduced to Computational Diffie-Hellman
  - Computational Diffie-Hellman can be reduced to Discrete Logarithm
- Computational Diffie-Hellman can be used to decrypt El Gamal ciphertext and vice versa

Diffie-Hellman Key Exchange



- Based on the difficulty of computational Diffie-Hellman
- Works also in extension Galois fields: $GF(p^q)$, ...

- How can we re-encrypt using El Gamal scheme without knowing the private key?