Let us now look at implementing graph algorithms in MapReduce.

Why Graphs?
- Discussion is based on the book and slides by Jimmy Lin and Chris Dyer
- Analyze hyperlink structure of the Web
- Social networks
  - Facebook friendships, Twitter followers, email flows, phone call patterns
- Transportation networks
  - Roads, bus routes, flights
- Interactions between genes, proteins, etc.

What is a Graph?
- \( G = (V, E) \)
  - \( V \): set of vertices (nodes)
  - \( E \): set of edges (links), \( E \subseteq V \times V \)
- Edges can be directed or undirected
- Graph might have cycles or not (acyclic graph)
- Nodes and edges can be annotated
  - E.g., social network: node has demographic information like age; edge has type of relationship like friend or family

Graph Problems
- Graph search and path planning
  - Find driving directions from A to B
  - Recommend possible friends in social network
  - How to route IP packets or delivery trucks
- Graph clustering
  - Identify communities in social networks
  - Partition large graph to parallelize graph processing
- Minimum spanning trees
  - Connected graph of minimum total edge weight

More Graph Problems
- Bipartite graph matching
  - Match nodes on “left” with nodes on “right” side
  - E.g., match job seekers and employers, singles looking for dates, papers with reviewers
- Maximum flow
  - Maximum traffic between source and sink
  - E.g., optimize transportation networks
- Finding “special” nodes
  - E.g., disease hubs, leader of a community, people with influence

Graph Representations
- Usually one of these two:
  - Adjacency matrix
  - Adjacency list
### Adjacency Matrix

- Matrix $M$ of size $|N| \times |N|$.
  - Entry $M(i,j)$ contains weight of edge from node $i$ to node $j$; 0 if no edge.
  
  $\begin{bmatrix}
  1 & 2 & 3 & 4 \\
  1 & 0 & 1 & 0 \\
  2 & 1 & 0 & 1 \\
  3 & 1 & 0 & 0 \\
  4 & 1 & 0 & 1 \\
  \end{bmatrix}$

Example source: Jimmy Lin

### Properties

- **Advantages**
  - Easy to manipulate with linear algebra.
  - Operation on outlinks and inlinks corresponds to iteration over rows and columns.

- **Disadvantage**
  - Huge space overhead for sparse matrix.
  - E.g., Facebook friendship graph.

### Adjacency List

- Compact row-wise representation of matrix.

  $\begin{bmatrix}
  1 & 2 & 3 & 4 \\
  1 & 0 & 1 & 0 \\
  2 & 1 & 0 & 1 \\
  3 & 1 & 0 & 0 \\
  4 & 1 & 0 & 1 \\
  \end{bmatrix}$

  - $1$: 2, 4
  - $2$: 1, 3, 4
  - $3$: 1
  - $4$: 1, 3

Example source: Jimmy Lin

### Properties

- **Advantages**
  - More space-efficient.
  - Still easy to compute over outlinks for each node.

- **Disadvantage**
  - Difficult to compute over inlinks for each node.

- Note: remember inverse Web graph discussion.

### Parallel Breadth-First Search

- **Case study:** single-source shortest path problem.
  - Find the shortest path from a source node $s$ to all other nodes in the graph.

- **For non-negative edge weights,** Dijkstra’s algorithm is the classic sequential solution.
  - Initialize distance $d(s)=0$, all others to $\infty$.
  - Maintain priority queue of nodes sorted by distance.
  - Remove first node $u$ from queue and update $d(v)$ for each node $v$ in adjacency list of $u$ if (1) $v$ is in queue and (2) $d(v) > d(u) + \text{weight}(u,v)$.

### Dijkstra’s Algorithm Example

Example from Jimmy Lin's presentation
Parallel Single-Source Shortest Path

- Priority queue is core element of Dijkstra’s algorithm
  - No global shared data structure in MapReduce
- Dijkstra’s algorithm proceeds sequentially, node by node
  - Taking non-min node could affect correctness of algorithm
- Solution: perform parallel breadth-first search
Parallel Breadth-First Search

• Start at source s
• In first round, find all nodes reachable in one hop from s
• In second round, find all nodes reachable in two hops from s, and so on
• Keep track of min distance for each node
  — Also record corresponding path
• Iterations stop when no shorter path possible

MapReduce Code: Single Iteration

```java
map(nid n, node N) // N stores node's current min distance and adjacency list
d = N.distance
emit(nid n, N) // Pass along graph structure
for all nid m in N.adjacencyList do
  emit(nid m, d + w(n,m)) // Emit distances to reachable nodes

reduce(nid m, [d1,d2,...]) // Recover graph structure
  dMin = ∞; M = ∅
  for all d in [d1,d2,...] do
    if isNode(d) then
      M = d // Update node's shortest distance
    else if d < dMin then
      dMin = d
      M.distance = dMin // Look for min distance in list
  emit(nid m, node M)
```

Overall Algorithm

• Need driver program to control the iterations
• Initialization: SourceNode.distance = 0, all others have distance = ∞
• When to stop iterating?
  • If all edges have weight 1, can stop as soon as no node has ∞ distance any more
    — Can detect this with Hadoop counter
  • Number of iterations depends on graph diameter
    — In practice, many networks show the small-world phenomenon, e.g., six degrees of separation

Dealing With Diverse Edge Weights

• “Detour” path can be shorter than “direct” connection, hence cannot stop as soon as all node distances are finite
• Stop when no node's shortest distance changes any more
  — Can be detected with Hadoop counter
  — Worst case: |N| iterations

MapReduce Algorithm Analysis

• Brute-force approach that performs many irrelevant computations
  — Computes distances for nodes that still have infinity distance
  — Repeats previous computations inside “search frontier”
• Dijkstra’s algorithm only explores the search frontier, but needs the priority queue
Typical Graph Processing in MapReduce

- Graph represented by adjacency list per node, plus extra node data
- Map works on a single node $u$
  - Node $u$’s local state and links only
- Node $v$ in $u$’s adjacency list is intermediate key
  - Passes results of computation along outgoing edges
- Reduce combines partial results for each destination node
- Map also passes graph itself to reducers
- Driver program controls execution of iterations

PageRank Introduction

- Popularized by Google for evaluating the quality of a Web page
- Based on random Web surfer model
  - Web surfer can reach a page by jumping to it or by following the link from another page pointing to it
  - Modeled as random process
- Intuition: important pages are linked from many other (important) pages
  - Goal: find pages with greatest probability of access

PageRank Definition

- PageRank of page $n$:
  - $P(n) = \alpha \frac{1}{|V|} + (1 - \alpha) \sum_{m \in L(n)} \frac{P(m)}{C(m)}$
  - $|V|$ is number of pages (nodes)
  - $\alpha$ is probability of random jump
  - $L(n)$ is the set of pages linking to $n$
  - $P(m)$ is $m$’s PageRank
  - $C(m)$ is $m$’s out-degree
- Definition is recursive
  - Compute by iterating until convergence (fixpoint)

Computing PageRank

- Similar to BFS for shortest path
- Computing $P(n)$ only requires $P(m)$ and $C(m)$ for all pages linking to $n$
  - During iteration, distribute $P(m)$ evenly over outlinks
  - Then add contributions over all of $n$’s inlinks
- Initialization: any probability distribution over the nodes

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PageRank Example
PageRank in MapReduce

MapReduce Code

map(nid n, node N)  // N stores node’s current PageRank and adjacency list
p = N.pageRank / |N.adjacencyList|  // Pass along graph structure
emit(nid n, N)  // Pass PageRank mass to neighbors

for all nid m in N.adjacencyList do
emit(nid m, p)  // Pass along graph structure

reduce(nid m, [p1,p2,...])
s = 0;
M = \emptyset
for all p in [p1,p2,...] do
if isNode(p) then
    M = p  // Recover graph structure
else
    s += p  // Sum incoming PageRank contributions
M.pageRank = \alpha/|V| + (1-\alpha) \cdot s
emit(nid m, node M)

Dangling Nodes

- Consider node x with no outgoing links
  - P(x) is not passed to any other node, hence gets “lost” in the Map phase
- Need to correct for the missing probability mass
  - Model: assume dangling page links to all pages
  - Mathematically equivalent to
    \[ P(n) = \frac{1}{|V|} + (1 - \alpha) \left( \frac{\delta}{|V|} + \sum_{m \in L(n)} \frac{P(m)}{C(m)} \right) \]
  - \delta: missing PageRank mass due to dangling nodes

PageRank with Dangling Nodes

- Challenge: need \delta, which is the sum over the current page ranks of dangling nodes
  - MR-job1: compute \delta
  - MR-job2: compute new PageRank using \delta
- Alternative computations?
  - Order inversion pattern to make sure \delta is available in all reduce tasks

Number of Iterations

- PageRank computation iterates until convergence
  - PageRank of all nodes no longer changes (or is within small tolerance)
  - Needs to be checked by driver
- Original PageRank paper: 52 iterations until convergence on graph with 322 million edges
  - Highly dependent on data properties

General Graph Processing Issues

- Sequential algorithms often use global data structure for efficiency
- In MapReduce with adjacency list representation, information can only be passed locally to or from direct neighbors
  - But can pre-compute other data structures, e.g., two-hop neighbors
- Presented algorithms have Map output of O(#edges), which works well for sparse graphs
General Graph Processing Issues

• Partitioning of graph into chunks strongly affects effectiveness of combiners
  – Often best to keep well-connected components together

• Numerical stability for large graphs
  – PageRank of individual page might be so small that it underflows standard floating point representation
  – Can work with logarithm-transformed numbers instead