Theory of Side-Choosing Games to Create and Disseminate Knowledge in Formal Science Domains

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Abstract

How do we create knowledge? We use the language of side-choosing games to instruct humans about which knowledge needs to be created, i.e., which problem needs solving. The language of side-choosing games becomes the domain-specific language for the human computation system. Our: side-choosing game (SCG) language uses a programming language to express game states and their transitions.

What is unique about our approach to create knowledge? To support a claim, a game is played. Playing a game only involves cleverly assigning values to variables and not knowledge about a formal proof system to support the claim. The side-choosing game approach supports a low key approach to defending a claim and is suitable for many skill levels. "Amateur Scientists" can help to gain insights into claims before they are passed on to the expert scientists.

How do we disseminate knowledge? Indirectly through curiosity! If a player loses a game although they predicted that they would win it, they become curious. They try to acquire the knowledge that the winning player has.

What is our theory: We study how to map side-choosing game results into ranking relations between players with guaranteed and desirables properties. We borrow ideas from social choice theory instead of using heuristics. Our theory informs the design of socio-technical systems for problem solving and teaching.

Our main contributions are (1) the concept of a side-choosing game with an illustration of its usefulness, (2) a representation theorem for an axiomatic treatment of ranking relations for side-choosing games, and (3) a meritocracy theorem showing that quasi-perfect players will be top-ranked iff the critical property of collusion-resistance holds.

1 Introduction

A side-choosing game is about a claim C which has an associated two-player, win-lose game $Game(C)$ between a proponent of C and an opponent of C such that a win of the proponent is an indication that C is true and a win of the opponent is an indication that C is false. $Game(C)$ does not allow for draws. If a proponent of C consistently wins against opponents, the proponent is said to have a winning strategy for C.

The first move in a side-choosing game is to choose a side for C: proponent or opponent. Player $x(y)$ chooses side $d(x)(d(y))$ (d = design-time). The side-choosing move may be simultaneous or sequential. For simplicity, we assume it is simultaneous. The side-choosing game includes an agreement algorithm that maps $x, y, d(x), d(y)$ into a set of games to be played where the design-time choices have been mapped into run-time choices $r(x), r(y)$ so that $r(x) \neq r(y)$. This requires that at most one of the players will be forced because it might be that $d(x) = d(y)$. A forced player is also called a devil's advocate especially when we think of a side-choosing game as a model for debates.

What is important to our theory is that for a claim C we get a table of game results ($winner = x, loser = y, d(x), d(y), r(x), r(y)$) for the games played between x and y. We simplify such a row to ($winner = x, loser = y, forced = z$), where z is either x, y or 0. We assume that the Players are $1, 2, 3, \ldots$. $z = 0$ means that none of the players is forced. The reason why this simplification works will be given in section ??.

GRID	Winner	Loser	Forced
gr1	1	2	1
gr2	1	2	1

Table 1: A Table T of game results

1.1 Players and Game Results

Let P be the set of all the players involved in the competition. Each game result has three columns corresponding to the winner, loser and the player forced to choose a side, if any. To represent a table T of game results we use a unique identifier for each row, called GRID. This guarantees that T will have no duplicates. The table 1 represents two game results where 1 played against 2 and where 1 won although it was forced. The game result tables can grow to any size as we allow the same players to play again. Of course, the game history for gr1 and gr2 might be very different.
$GR(P)$ is the set of the all possible game results without the unique identifier GRID. $GR(P)$ contains $n \cdot (n-1) \cdot 3$ rows where n is the number of players in P. $G(P)$ is the set of all possible game results with unique identifiers.

1.2 Examples of Side-Choosing Games

The definition of a side-choosing game given above is abstract and only useful if we can give several interesting concrete versions.

- **Combinatorial Games** We choose a combinatorial game [5] and a position pos. The claim is: the position pos is a winning position.
- **Semantic Games** We choose a logic which supports semantic games. I.e., each sentence in the logic is mapped to a game between proponent and opponent so that the claim is true iff the proponent has a winning strategy. The sentences are interpreted in some structure. Most logics have semantic games. Some prominent examples are first and higher-order logics and independence-friendly logic [11].

  Semantic games are a huge application domain for side-choosing games and we arrived at side-choosing games through the study of semantic games [7].

  The connection between proofs and winning strategies is an active topic in logic [3]. One of the attractions of side-choosing games is that you don't need a proof for a claim to perform well in the side-choosing game for the claim. Side-choosing games are more easily accessible than formal proofs.

1.3 Applications of Side-Choosing Games

Our study of side-choosing games is motivated by their potential to organize problem-solving competitions and by their successful use in education at Northeastern University. We believe they are the foundation for platforms like TopCoder or Kaggle or scientific human-computation tools like Fold-It [4].

- **Education in Formal Sciences** Our favorite way of summarizing learning objectives for a formal science domain is to say that learners must demonstrate the skill of judging claims in the domain, choosing their side on the claim and then defending their side choice through game play against other students. The resulting peer-teaching and peer-grading is very attractive.

  A claim is representing a lab in which students learn and is chosen in such a way that solving the problem requires skills that students should have.

- **Improving Evaluation in Problem-Solving Competitions for Computational Problems** A significant advantage of our approach is that the evaluation of solutions is done by peers and not the competition organizer. The competition organizer only acts in a role as referee. Instead of static benchmarks, dynamic benchmarks are developed through game play.

The quality of the solutions produced depends on the skills of the participants who might not be motivated or not have the knowledge necessary to solve the problem. To attract strong participants either money or fame has to be given; a common theme in human computation.

1.4 Related Work

[7] Defines an important class of binary side-choosing games called semantic games and relates it to the foundations of logic.

In [10], Rubinstein provides an axiomatic treatment of tournament ranking functions that bears some resemblance to ours. Rubinstein's treatment was developed in a primitive framework where "beating functions" are restricted to complete, asymmetric relations. Rubinstein showed that the points system, in which only the winner is rewarded with a single point is *completely* characterized by the following three *natural* axioms:

- anonymity which means that the ranks are independent of the names of participants,
- positive responsiveness to the winning relation which means that changing the results of a participant p from a loss to a win, guarantees that p would have a better rank than all other participants that used to have the same rank as p, and
- Independence of Irrelevant Matches (IIM) which means that the relative ranking of two participants is independent of those matches in which neither is involved.

Our LFB axiom is, in some sense, at least as strong as Rubinstein's IIM because, according to LFB, the relative rank of some participant p_x w.r.t. another participant p_y cannot be worsened by games that p_x does not participate in nor can it be improved by games that p_y does not participate in.

[3] discusses the connection between winning strategies for semantic games and proofs. A recursive winning strategy for a semantic game of a sentence is a constructive proof of that sentence. They introduce the notion of CGTS-truth (computable game-theoretical semantics truth): a sentence ϕ is CGTS-true on a recursive model M exactly when there is a computable winning strategy for Verifier in the semantical game played with ϕ on M.

They focus on the special case of Peano Arithmetic (PA). They investigate the following questions

- From proofs to winning strategies Do proofs in PA yield CGTS-truth?
- From winning strategies to proofs Can the CGTS-truth of a sentence be interpreted as a proof?

Side-choosing games with backward moves are important in the study of those questions. The backward moves allow for many more winning strategies.

Rating methods can be used to rank tournament participants. There is a vast body of literature on the topic of *heuristic* [2] rating methods aiming to estimate the skill level of participants such as the Elo [6] rating method. [8] gives a recent comprehensive overview of rating methods used in sports tournaments. Our work differs from this vast body of literature in two important aspects. First, our axioms and ranking method are the first to be developed for an extended framework that we developed specifically to capture some of the peculiarities of side-choosing game tournaments such as forcing. Second, our work is the *first* to be concerned with collusion resistance.

An early version of SCG, then called Scientific Community Game, was published in [9] "Raghav: please make this work: bibtex file seems to have duplications?" .

This paper is based on Ahmed Abdelmeged's dissertation [1]. The dissertation is based on semantic games and does not explicitly define side-choosing games. However, the proof of the representation theorem does not rely on a specific logic. Therefore, we introduced side-choosing games in this paper to have an appropriate context for formulating and proving the representation theorem. The proofs in this paper have been simplified through the systematic use of monotonicity constraints.

2 Model

2.1 Basic Predicates and Operations

$$(\forall p_x \in P, \forall r \in G(P))[\mathit{particpant}(p_x, r) \Leftrightarrow p_x \text{ is a participant in the game } r]$$
$$(\forall p_x \in P, \forall r \in G(P))[\mathit{win}(p_x, r) \Leftrightarrow p_x \text{ won the game } r]$$
$$(\forall p_x \in P, \forall r \in G(P))[\mathit{loss}(p_x, r) \Leftrightarrow p_x \text{ lost the game } r]$$
$$(\forall p_x \in P, \forall r \in G(P))[\mathit{forced}(p_x, r) \Leftrightarrow p_x \text{ is forced to choose a side}]$$
$$(\forall p_x \in P, \forall r \in G(P))[\neg \mathit{control}(p_x, r) \Leftrightarrow \neg \mathit{participant}(p_x, r) \vee (\mathit{loss}(p_x, r) \wedge \mathit{forced}(p_x, r))]$$
$$(\forall p_x \in P, \forall r \in G(P))[\mathit{fault}(p_x, r) \Leftrightarrow \mathit{loss}(p_x, r) \wedge \neg \mathit{forced}(p_x, r)]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[\mathit{wf}^T(p_x) = \text{the win count of } p_x \text{ in } T \text{ in a forced position}]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[\mathit{wu}^T(p_x) = \text{the win count of } p_x \text{ in } T \text{ in an unforced position}]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[\mathit{lf}^T(p_x) = \text{the loss count of } p_x \text{ in } T \text{ in a forced position}]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[\mathit{lu}^T(p_x) = \text{the loss count of } p_x \text{ in } T \text{ in an unforced position}]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[\mathit{np}^T(p_x) = \text{the number of games in } T \text{ where } p_x \text{ was not a participant}]$$

3 Ranking

In this section, we discuss ranking the players.

3.1 Ranking Relation

We define a preorder $\preceq_U^T$ called the weakly better relation $\forall T \subseteq G$ based on the scoring function $U : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}$. For convenience, we drop the subscript and refer to it simply as $\preceq^T$.
We want to assign each player a score solely based on the players' demonstration of ability. We use the 4 statistics based on wins and losses of the player calculate the players' score. We formally define the ranking relation as,

$$\begin{aligned}(\forall p_x, p_y \in P, \forall T \subseteq G(P) : \ (U : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}))[p_x \preceq^T p_y \Leftrightarrow \\ U(\mathit{wf}^T(p_x), \mathit{wu}^T(p_x), \mathit{wf}^T(p_x), \mathit{lu}^T(p_x)) \leq U(\mathit{wf}^T(p_y), \mathit{wu}^T(p_y), \mathit{wu}^T(p_y), \mathit{lu}^T(p_y))]\end{aligned} \tag{1}$$

We want the ranking relation to have the following properties:

- NNEW: $(\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T \ : \ \mathit{win}(p_x, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$
- NPEL: $(\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T \ : \ \mathit{loss}(p_y, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$
- CR: $(\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T \ : \ \neg \mathit{control}(p_x, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$

3.2 Universal Domain

From equation 1, it is clear that for every logically possible game result table T, we have a valid preorder. This implies that our ranking relaton satisfies the Universal Domain property.

3.3 Anonymity

From equation 1 it is clear that the scoring function ignores the identity of the player in calculating the score. Hence, the ranking relation $\preceq^T$ is unaffected by changing labels and therefore anonymous.

3.4 Montonicity of U and Notation

At the heart, NNEW and NPEL are monotonicity conditions. As we score the player solely based on the players' wins and losses, NNEW and NPEL imply that the function U is monotonic. One interesting thing about the parameters of U for a particular player is that when we add a new game to the existing game result table T, atmost one parameter increments. This allows us to use a notation that mimics the partial differential operator. The notation will come handy to prove some interesting results.

$$\partial_x U \geq 0 \;:\; U \text{ is monotonically non-decreasing on the parameter } x$$
$$\partial_x U > 0 \;:\; U \text{ is monotonically increasing on the parameter } x$$
$$\partial_x U \leq 0 \;:\; U \text{ is monotonically non-increasing on the parameter } x$$
$$\partial_x U <: \; U \text{ is monotonically decreasing on the parameter } x$$
$$\partial_x U = 0 \;:\; U \text{ is indifferent on the paramenter } x$$

4 Properties of Ranking Relations

In this section, we formulate properties in predicate logic and dervice their equilvalent monotonicity constraints.

4.1 Collusion Resistance (CR)

$$\text{CR: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T \;:\; \neg\mathtt{control}(p_x, r))[p_x \preceq^{T} p_y \Rightarrow p_x \preceq^{T\cup\{r\}} p_y]$$

Here we have 2 cases, game results where p_x did not particpate and game results where p_x lost when forced.For the first case, p_y may have won or lost in a forced or unforced position against some third player p_z.
Let us consider the case where p_y wins over p_z in a forced position, we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y)+1, wu^T(p_y), lf^T(p_y), lu^T(p_y)) \tag{2}$$

From equations 1 and 2, we get the monotonicity constraint,

$$\partial_{wf} U \geq 0 \tag{3}$$

Let us consider the case where p_y wins over p_z in an unforced position, we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y)+1, lf^T(p_y), lu^T(p_y)) \tag{4}$$

From equations 1 and 4, we get the monotonicity constraint,

$$\partial_{wu} U \geq 0 \tag{5}$$

Let us consider the case where p_y loses over p_z in a forced position, we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y)+1, lu^T(p_y)) \tag{6}$$

From equations 1 and 6, we get the monotonicity constraint,

$$\partial_{lf} U \geq 0 \tag{7}$$

Let the consider the case where p_y loses over p_z in an unforced position, we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y)+1) \tag{8}$$

From equations 1 and 8, we get the montonicity constraint,

$$\partial_{lu} U \geq 0 \tag{9}$$

Now we consider game results where p_x was forced to lose against some third player p_z,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x) + 1, lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y)) \tag{10}$$

From equations 1 and 10, we get the motonicity constraint,

$$\partial_{lf} U \leq 0 \tag{11}$$

Now, CR can be summarised in terms of monotonicity constraints as,

$$\partial_{wf} U \geq 0 \wedge \partial_{wu} U \geq 0 \wedge \partial_{lf} U = 0 \wedge \partial_{lu} U \geq 0 \tag{12}$$

4.2 Non Negative Effect of Winning (NNEW)

$$\text{NNEW: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : win(p_x, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

Let us consider a game result r where p_x won against a third player p_z. p_x could have won either in a forced or unforced position.
Let us consider the case where p_x wins over p_z in a forced position, we have,

$$U(wf^T(p_x) + 1, wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y)) \tag{13}$$

From equations 1 and 13, we get the monotnicity constaint,

$$\partial_{wf} U \leq 0 \tag{14}$$

Let us consider the case where p_x wins over p_z in an unforced position, we have:

$$U(wf^T(p_x), wu^T(p_x) + 1, lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y)) \tag{15}$$

From equations 1 and 15, we get the monotnicity constaint,

$$\partial_{wu} U \leq 0 \tag{16}$$

Summarising the monotonicty constraints, we have,

$$\partial_{wf} U \leq 0 \wedge \partial_{wu} U \leq 0 \tag{17}$$

4.3 Non Positive Effect of Losing (NPEL)

$$\text{NPEL: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : loss(p_y, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

Let us consider a game result r where p_y lost against a third player p_z. p_y could have lost either in a forced or unforced position.
Let us consider the case where p_y loses over p_z in a forced position, we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y) + 1, lu^T(p_y)) \tag{18}$$

From equations 1 and 18, we get the monotnicity constaint,

$$\partial_{lf} U \geq 0 \tag{19}$$

Let us consider the case where p_y loses over p_z in an unforced position, we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y) + 1) \tag{20}$$

From equations 1 and 20, we get the monotnicity constaint,

$$\partial_{lu} U \geq 0 \tag{21}$$

Summarising the monotonicty constraints, we have,

$$\partial_{lf} U \geq 0 \wedge \partial_{lu} U \geq 0 \tag{22}$$

5 Discussion

As we want the ranking relation to satisfy all the three properties NNEW, NPEL and CR, from equations 12, 17 and 22, we get the monotonicity constraints,

$$\partial_{wf}U = 0 \wedge \partial_{wu}U = 0 \wedge \partial_{lf}U = 0 \wedge \partial_{lu}U \geq 0 \tag{23}$$

This tells us that the scoring function should be monotonically non-decreasing on faults and indifferent on other parameters. We call the ranking relation that uses a scoring function that satisfies equation 23 as Local Fault Based (LFB). The monotonicity constraints in equation 23 can be easily reformulated in predicate logic. The reformulation is given in section 5.1. Also, we define a so far not covered property called Independent of Irrelevant Games (IIG) in section 5.2. IIG is well-known in the Social Choice literature

5.1 Local Fault Based (LFB)

$$\text{LFB:}(\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \neg \mathtt{fault}(p_x, r) \wedge \neg \mathtt{fault}(p_y, r))[p_x \preceq^T p_y \Leftrightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

The monotonicity constraints are the same as equation 23. Repeating here, we have,

$$\partial_{wf}U = 0 \wedge \partial_{wu}U = 0 \wedge \partial_{lf}U = 0 \wedge \partial_{lu}U \geq 0 \tag{24}$$

5.2 Independent of Irrelevant Games (IIG)

$$\text{IIG: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \neg \mathtt{participant}(p_x, r) \wedge \neg \mathtt{participant}(p_y, r))[p_x \preceq^T p_y \Leftrightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

This property just means that relative ranking does not depend on the games where neither p_x nor p_y has participated. From the definitions of LFB and IIG, it is simple to deduce that,

$$\text{LFB} \Rightarrow \text{IIG} \tag{25}$$

But,

$$\text{IIG} \not\Rightarrow \text{LFB} \tag{26}$$

This leads us to the conclusion that LFB is at least as strong as IIG.

6 Representation theorem

In this section, we prove an interesting result that will aid us in exploring the appropriate set of scoring functions. Let us first summarise the properties in monotonicity constraints

6.1 Summary of Monotonicity Constraints

$$\text{CR: } \partial_{wf}U \geq 0 \wedge \partial_{wu}U \geq 0 \wedge \partial_{lf}U = 0 \wedge \partial_{lu}U \geq 0$$
$$\text{LFB: } \partial_{wf}U = 0 \wedge \partial_{wu}U = 0 \wedge \partial_{lf}U = 0 \wedge \partial_{lu}U \geq 0$$
$$\text{NNEW: } \partial_{wf}U \leq 0 \wedge \partial_{wu}U \leq 0$$
$$\text{NPEL: } \partial_{lf}U \geq 0 \wedge \partial_{lu}U \geq 0$$

6.2 The Theorem : NNEW $\wedge$ NPEL $\Rightarrow$ (CR $\Leftrightarrow$ LFB)

It is easy to derive the following relations from the monotonicity constraints summarised above,

$$\text{NNEW} \wedge \text{CR} \Rightarrow \text{LFB} \tag{27}$$

$$\text{NPEL} \wedge \text{LFB} \Rightarrow \text{CR} \tag{28}$$

From equations 27 and 28, we deduce that,

$$\text{NNEW} \wedge \text{NPEL} \Rightarrow (\text{CR} \Leftrightarrow \text{LFB}) \tag{29}$$

7 Ramifications of the Representation Theorem

7.1 IIG and Representation theorem

We discuss if non participation as a parameter in the scoring function U would have any effect on the monotonicity constraints we have derived so far. For that, we have to derive the monotonicity constraint for IIG.
The number of games p_x has not participated in is defined as,

$$np^T(p_x) = |T| - wf^T(p_x) - wu^T(p_x) - lf^T(p_x) - lu^T(p_x) \tag{30}$$

From equations 1 and 30, it is clear that we can calculate $np^T(p_x)$ from $|T|$ and existing 4 parameters of U. To account for non participation, we can redefine the ranking relation as,

$$\begin{aligned}(\forall p_x, p_y \in P, T \subseteq G(P) : (U : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}))[p_x \preceq^T p_y \Leftrightarrow \\ U(wf^T(p_x), wu^T(p_x), wf^T(p_x), lu^T(p_x), |T|) \le U(wf^T(p_y), wu^T(p_y), wu^T(p_y), lu^T(p_y), |T|)]\end{aligned} \tag{31}$$

Now let us add a game result r where neither p_x nor p_y participate. From the definition of IIG we have,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x), |T|+1) \le U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y), |T|+1) \tag{32}$$

For equations 31 and 32 to hold together, we need that equations 33 and 34 to hold. First, we focus on p_x and we need that,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x), |T|+1) \le U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x), |T|) \tag{33}$$

Now, we focus on p_y and we need that,

$$U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y), |T|+1) \ge U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y), |T|) \tag{34}$$

This gives us the monotonicity constraints,

$$\partial_{|T|}U \le 0 \wedge \partial_{|T|}U \ge 0 \tag{35}$$

This is the same as,

$$\partial_{|T|}U = 0 \tag{36}$$

From equation 36, it is clear that the IIG is the same as the scoring function U being indifferent to $|T|$.
From equations 25 and 29, we have,

$$\text{CR} \Rightarrow \text{IIG} \tag{37}$$

From equations 36 and 37, it is clear that when CR holds, our initial formulation of not choosing $|T|$ as a parameter in U is reasonable. This means that we need not revise the monotonicity constraints in section 6.1.

7.2 Incentive Compatibility (IC)

$$\text{IC: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \mathtt{fault}(p_y, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

This is equivalent to incentivising not making faults. Let us first consider the game where p_y made a fault with a third player p_z. In that case,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y) + 1) \tag{38}$$

From equations 1 and 38, we get the monotnicity constraint,

$$\partial_{lu} U \geq 0 \tag{39}$$

Now, let us consider the game where p_y made a fault in a game with p_x. In this case, there are 2 distinct possibilties. The first is p_x won the game and was forced, we have

$$U(wf^T(p_x) + 1, wu^T(p_x), lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y) + 1) \tag{40}$$

From equations 38 and 40, we get the monotnicity constraint,

$$\partial_{wf} U \leq 0 \tag{41}$$

Let us consider the case where p_x won in an unforced position.

$$U(wf^T(p_x), wu^T(p_x) + 1, lf^T(p_x), lu^T(p_x)) \leq U(wf^T(p_y), wu^T(p_y), lf^T(p_y), lu^T(p_y) + 1) \tag{42}$$

From equations 38 and 42, we get the monotnicity constraint,

$$\partial_{wu} U \leq 0 \tag{43}$$

Summarising the monotonicty constraints for IC, we have,

$$\partial_{wf} U \leq 0 \wedge \partial_{wu} U \leq 0 \wedge \partial_{lu} U \geq 0 \tag{44}$$

From equations 24 and 44, we have that,

$$\text{LFB} \Rightarrow \text{IC} \tag{45}$$

From equations 17 and 44, we have that,

$$\text{IC} \Rightarrow \text{NNEW} \tag{46}$$

7.3 Example Scoring functions that are LFB

The representation theorem motivates us to explore the set of scoring functions that are LFB. Here, we list a set of 3 representative examples. It is useful to have families of LFB ranking functions to choose an appropriate member for a given competition.

7.3.1 Everyone get the same score

The simplest example of a LFB scoring sunction is where every player gets the same score.

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) = 0 \tag{47}$$

It is easy to see that this scoring function satisfies the following monotonicity constraints,

$$\partial_{wf} U = 0 \wedge \partial_{wu} U = 0 \wedge \partial_{lf} U = 0 \wedge \partial_{lu} U = 0 \tag{48}$$

From equations 24 and 48, it is clear that this scoring function is LFB.

7.3.2 Regular fault counting

Another example would be to count faults.

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) = lu^T(p_x) \tag{49}$$

This scoring function satisfies the following monotonicity constraints,

$$\partial_{wf}U = 0 \wedge \partial_{wu}U = 0 \wedge \partial_{lf}U = 0 \wedge \partial_{lu}U > 0 \tag{50}$$

From equations 24 and 50, it is clear that this scoring function is LFB.

7.3.3 Weighted fault counting

Now, we explore a more general version of fault counting. We consider positive weights assignment to fault classes where:

- The winner is a forced opponent, weakness in proponent (α fault): Weight = α.
- The winner is a forced proponent, weakness in opponent (β fault): Weight = β.
- The winner is a non-forced opponent, weakness in proponent (γ fault): Weight = γ
- The winner is a non-forced proponent, weakness in opponent (δ fault): Weight = δ

At this juncture, we need to define a few functions,

$$(\forall p_x \in P, \forall T \subseteq G(P))[lu^T_\alpha(p_x) = \text{the number of } \alpha \text{ faults of } p_x \text{ in } T]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[lu^T_\beta(p_x) = \text{the number of } \beta \text{ faults of } p_x \text{ in } T]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[lu^T_\gamma(p_x) = \text{the number of } \gamma \text{ faults of } p_x \text{ in } T]$$
$$(\forall p_x \in P, \forall T \subseteq G(P))[lu^T_\delta(p_x) = \text{the number of } \delta \text{ faults of } p_x \text{ in } T]$$

It is also very clear that,

$$lu^T(p_x) = lu^T_\alpha(p_x) + lu^T_\beta(p_x) + lu^T_\gamma(p_x) + lu^T_\delta(p_x) \tag{51}$$

Now, we define the scoring function as,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) = \alpha \cdot lu^T_\alpha(p_x) + \beta \cdot lu^T_\beta(p_x) + \gamma \cdot lu^T_\gamma(p_x) + \delta \cdot lu^T_\delta(p_x) \tag{52}$$

$\forall \alpha, \beta, \gamma, \delta \in \mathbb{R}^+$, it is easy to observe that this scoring function satisfies the following monotonicity constraints,

$$\partial_{wf}U = 0 \wedge \partial_{wu}U = 0 \wedge \partial_{lf}U = 0 \wedge \partial_{lu}U > 0 \tag{53}$$

From equations 24 and 53, it is clear that this scoring function is LFB.

8 Top Ranking and Meritocracy

In this section we show a desirable implication of cullusion-resistance. We introduce the concept of quasi-perfect player and show that quasi-perfect players are top-ranked iff the ranking relation is collusion-resistant. The quasi-perfect players are the players which have the minimum number of faults among all players. A quasi-perfect player may defend false claims and refute true claims and make "wrong" side choices. This differs from a perfect player who always makes the correct side-choices and successfully defends her choice.

To talk about top ranking, we need to first define the strictly better relation $\prec^T$.

$$\begin{aligned}&(\forall p_x, p_y \in P, \forall T \subseteq G(P) : (U : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}))[p_x \prec^T p_y \Leftrightarrow \\ &\quad U(wf^T(p_x), wu^T(p_x), wf^T(p_x), lu^T(p_x)) < U(wf^T(p_y), wu^T(p_y), wu^T(p_y), lu^T(p_y))]\end{aligned} \tag{54}$$

8.1 Top Ranking

A player p in a set of players P is said to be top-ranked $TR^T(p)$ in a set of game results T, if there is no player that is strictly better than p. More generally, it is defined in predicate logic as,

$$(\forall p, p_x \in P, \forall T \subseteq G(P))[TR^T(p) \Leftrightarrow \neg(p_x \prec^T p)] \tag{55}$$

8.2 Quasi Perfection

A quasi-perfect player for a set of players is a player who makes the least number of faults within the set of players. A perfect player always chooses the correct side and always wins the defense and therefore is quasi-perfect. Such a player makes no faults.

8.3 Meritocracy Theorem

To Ruiyang:

There is an issue with the proof that CR is necessary for quasi-perfect players to be top-ranked. A ranking relation R has the quasi-top property $quasiTop(P, R)$, for a set of players P if all quasi-perfect players in P are top-ranked. Can you show that if a ranking relation for P is not CR than it is not quasiTop, i.e.,

$$\forall P \forall R(P) : quasiTop(P, R(P)) \Rightarrow CR(P, R(P)).$$

I was looking for a constructive proof which constructs for a ranking-relation that is not CR a set of games where there exists a quasi-perfect player that is not top-ranked.

Raghav was suggesting that there is an other proof based on monotonicity constraints which shows

$$\forall P \forall R(P) : quasiTop(P, R(P)) \Rightarrow CR(P, R(P)).$$

Can you help us to complete the proof of the Meritocracy theorem?

- CR is necessary: If the ranking relation is not collusion-resistant (CR) but NNEW $\wedge$ NPEL, there exists a set of games where a quasi-perfect player is not top-ranked.
- CR is sufficient: NNEW $\wedge$ NPEL $\wedge$ CR imply that all quasi-perfect players are top-ranked.

Let us prove the first part of the theorem. First, we write down the monotonicty contraints for $\neg$CR and NNEW $\wedge$ NPEL. We have,

$$\neg\text{CR: } \partial_{wf}U < 0 \vee \partial_{wu}U < 0 \vee \partial_{lf}U > 0 \vee \partial_{lf}U < 0 \vee \partial_{lu}U \leq 0 \tag{56}$$

$$\text{NNEW} \wedge \text{NPEL: } \partial_{wf}U \leq 0 \wedge \partial_{wu}U \leq 0 \wedge \partial_{lu}U \geq 0 \wedge \partial_{lf}U \geq 0 \tag{57}$$

The equations 56 and 57 give us a clue into exploring a scoring function for our ranking relation that is $\neg$CR, but NNEW $\wedge$ NPEL. One proposal for such a scoring function is,

$$U(wf^T(p_x), wu^T(p_x), lf^T(p_x), lu^T(p_x)) = -(wf^T(p_x) + wu^T(p_x)) \tag{58}$$

With the scoring function as specified by equation 58 (which is basically counting the total number of wins for a particular player), consider the table 2 of game results. Here, player 1 is top ranked with the 4 wins and 2 faults. But, player 2, the quasi perfect player with no faults is not top ranked. This proves the first part of the meritocracy theorem.

To prove the second part of the theorem, we use the meritocracy theorem. From equation 29, we know that with NNEW $\wedge$NPEL, CR is the same as LFB. From the very definition of a quasi perfect player being the one with the least faults and LFB just being a scoring fucnction that is non decreasing on faults and being non responsive to other parameters, it is clear that the quasi perfect player will be top ranked.

GRID	Winner	Loser	Forced
gr1	1	2	2
gr2	1	3	-
gr3	1	2	2
gr4	2	1	-
gr5	2	3	2
gr6	2	3	-
gr7	3	1	-
gr8	1	3	1

Table 2: A Table T of game results for 3 players

9 Skype Call Summaries

9.1 Dec. 12, 2014

NNEW and NPEL imply monotonicity of Scoring function U.

Introduce independence of irrelevant games (IIG) as additional axiom. CR implies IIG.

Definition of independence of irrelevant games: The ranking of two players only depends on the games in which at least one participates.

What can we say about NNEW and NPEL and IIG? Table-size is irrelevant.

Is collusion possible? Yes.

Introduce collusion resistance. Provides a useful form of anonymity. Introduces another kind of row that is irrelevant: a player p who is forced and loses: such a row cannot lower the rank of p. Affinity to NNEW.

9.2 Dec. 16, 2014

Where is the Universal Domain property?

No neutrality: Side-choosing games don't support neutrality because it does not make sense to change side-choices (in analogy to changing votes).

Introduce two names for game results: with and without the game result identifier. GR(P): without game result identifier. G(P): with game result identifier.

There are two kinds of anonymity: scoring function/ranking relation based and non-anonymity to influence game results.

Introduce family of LFB ranking functions.

Add section 8 to discuss extensions: independence of table size.

In introduction discuss problem with sybils and collusion.

9.2.1 Short Proof: LFB implies CR

We provide a short proof

$$\text{CR: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \neg \mathtt{control}(p_x, r))[p_x \preceq^T p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

$$\text{LFB: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \neg \mathtt{fault}(p_x, r) \wedge \neg \mathtt{fault}(p_y, r))[p_x \preceq^T p_y \Leftrightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

We assume that LFB holds. if both p_x and p_y have not made a fault, there are two cases:

- Case 1

  Both don't participate. This implies that both did not make a fault and therefore both are not in control. From the LFB side we know that adding such a row does not change the ranking. Therefore CR is implied.

- Case 2 One player must have been forced because the game only has a winner and loser. The winner, by definition, does not make a fault. The loser does not make a fault iff it is forced. The player who is forced is not in control. From the LFB side we know that adding a row where none made a fault does not change the ranking. Therefore LFB implies CR for the second case.

It is interesting that this direction does not require NNEW and NPEL.

For the opposite direction we need the proof based on monotonicity constraints. But LFB implies CR holds unconditionally. This can also be seen directly from the proof of the representation theorem.

9.2.2 Dec. 23

- Logical relationships between properties.

  We have 4 Boolean properties: NNEW, NPEL, LFB, CR with the following relationships (1a) to (1d). This gives more information than only the representation theorem which is implied by (1c) and (1d).

  (1a) LFB implies NNEW

  (1b) LFB implies NPEL

  (1c) LFB implies CR

  (1d) (CR and NNEW and NPEL) implies LFB

  Representation Theorem: (NNEW and NPEL) implies (LFB iff CR) (implied by (1c) and (1d))

- Logical relationships involving IIG.

  CR implies IIG

  LFB implies IIG

  not (IIG implies LFB)

9.2.3 Dec. 28

Raghav uses the following to deincentivize faults:

$$\text{IC: } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \mathit{fault}(p_y, r))[p_x \preceq^{T} p_y \Rightarrow p_x \preceq^{T \cup \{r\}} p_y]$$

What if we use instead:

$$\text{IC': } (\forall p_x, p_y \in P, \forall T \subseteq G(P), \forall r \in G(P) \setminus T : \mathit{fault}(p_y, r))[p_x \preceq^{T} p_y \Rightarrow p_x \prec^{T \cup \{r\}} p_y]$$

Note that the last "weakly better" is replaced by "better." IC' more strongly deincentivzes faults. IC implies NNEW. IC' implies NNEW.

We decided to use the following **definition of quasi-perfect:** A quasi-perfect player for a group of players is a player who makes the least number of faults within the group. (A perfect player always chooses the correct side and always wins the defense and therefore is quasi-perfect. She makes 0 faults.)

Definition of top-ranked for a set of players Ps: TR(p in Ps)= There is no player p1 (p1 != p) that is strictly better than p.

Equivalent?: For all p1 in Ps s.t. p1 != p: p is weakly better than p1. top-ranked is defined in terms of the total preorder.

We say that p_x is strictly better than p_y if $p_y \not\preceq p_x$.

Meritoracy Theorem

Part 1: [CR is necessary]: If ranking function is not collusion-resistant (CR) but NNEW and NPEL, there exists a set of games where a quasi-perfect player is not top-ranked. Part 2: [CR is sufficient] NNEW and NPEL and CR imply that all quasi-perfect players are top-ranked.

See 7.3 for set of monotonicity constraints that should be helpful to construct the set of games.

The proof of part 2 relies on representation theorem. We know that LFB holds.

Space of LFB ranking functions Give 3 representative examples? a) count faults b) give different weights to faults: other forced/unforced c) give all the same score.

9.2.4 Dec. 29, 2014

Produce a **Venn diagram that summarizes the results**. Use www.lucidchart.com to conveniently draw it? Or just powerpoint?

We have 5 Boolean variables: NNEW,NPEL,CR,LFB,IC. If they are in general position, we have 32 different subsets. Because of the properties, such as (LFB implies CR), (IC implies NNEW), we have many fewer. How many? This venn diagrams should be acompanied by a legend which lists all properties we proved.

IC' implies IC.

not(LFB implies IC'). Counterexample: everybody gets same rank. Is LFB but not IC'.

Reviewing the definition of quasi-perfect: minimum number of faults (lu) and maximum number of wins when opponent is forced (waf = win against forced). A perfect player has lu=0 and waf=all. When no lu-events happen and all events are waf-events: quasi equilibrium. quasi equilibrium is equilibrium if players are perfect.

quasi-equilibrium: all players choose same side, only waf-events.

Table T, set of game results. T is at quasi-equilibrium if there are only waf events in T. A player p is quasi-perfect for T if p has made the minimum number of faults among all players in T.

```
W L F
1 2 2 win against forced event
1 2 1 not win against forced event
```

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