Rock-Paper-Scissors

A two person game.

Rules. At the count of three declare one of:

<table>
<thead>
<tr>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
</table>

Winner Selection. Identical selection is a draw. Otherwise:

- Rock beats Scissors
- Paper beats Rock
- Scissors beats Paper

Payoff Matrix. Payoffs are from row player to column player:

\[
A = \begin{bmatrix}
P & S & R \\
P & 0 & 1 & -1 \\
S & -1 & 0 & 1 \\
R & 1 & -1 & 0 \\
\end{bmatrix}
\]

Note: Any deterministic strategy employed by either player can be defeated systematically by the other player.
Two-Person Zero-Sum Games

Given: $m \times n$ matrix $A$.

- Row player (rowboy) selects a strategy $i \in \{1, \ldots, m\}$.
- Col player (colgirl) selects a strategy $j \in \{1, \ldots, n\}$.
- Rowboy pays colgirl $a_{ij}$ dollars.

Note: The rows of $A$ represent deterministic strategies for rowboy, while columns of $A$ represent deterministic strategies for colgirl.

Deterministic strategies can be bad.
Randomized Strategies.

- Suppose rowboy picks \( i \) with probability \( y_i \).
- Suppose colgirl picks \( j \) with probability \( x_j \).

Throughout, \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \) and \( y = [y_1 \ y_2 \ \cdots \ y_m]^T \) will denote **stochastic vectors**:

\[
x_j \geq 0, \quad j = 1, 2, \ldots, n \quad \sum_j x_j = 1.
\]

If rowboy uses random strategy \( y \) and colgirl uses \( x \), then *expected payoff* from rowboy to colgirl is

\[
\sum_i \sum_j y_i a_{ij} x_j = y^T A x
\]
Colgirl’s Analysis

Suppose colgirl were to adopt strategy $x$.

Then, rowboy’s best defense is to use $y$ that minimizes $y^T Ax$:

$$\min_y y^T Ax$$

And so colgirl should choose that $x$ which maximizes these possibilities:

$$\max_x \min_y y^T Ax$$
Solving Max-Min Problems as LPs

Inner optimization is easy:

\[
\min_y y^T Ax = \min_i e_i^T Ax
\]

(\(e_i\) denotes the vector that’s all zeros except for a one in the \(i\)-th position—that is, deterministic strategy \(i\)).

**Note:** Reduced a minimization over a *continuum* to one over a *finite set*.

We have:

\[
\max \left( \min_i e_i^T Ax \right)
\]

\[
\sum_j x_j = 1,
\]

\[
x_j \geq 0, \quad j = 1, 2, \ldots, n.
\]
Reduction to a Linear Programming Problem

Introduce a scalar variable $v$ representing the value of the inner minimization:

$$\begin{align*}
\max v \\
v &\leq e_i^T Ax, \quad i = 1, 2, \ldots, m, \\
\sum_j x_j &= 1, \\
x_j &\geq 0, \quad j = 1, 2, \ldots, n.
\end{align*}$$

Writing in pure matrix-vector notation:

$$\begin{align*}
\max v \\
v e - Ax &\leq 0 \\
e^T x &= 1 \\
x &\geq 0
\end{align*}$$

($e$ denotes the vector of all ones).
Finally, in Block Matrix Form

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix}^T \begin{bmatrix}
x \\
v
\end{bmatrix}
\]

\[
\begin{bmatrix}
-A & e \\
e^T & 0
\end{bmatrix} \begin{bmatrix}
x \\
v
\end{bmatrix} \leq \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[x \geq 0\]

\[v \text{ free}\]
Similarly, rowboy seeks $y^*$ attaining:

$$\min_y \max_x y^T Ax$$

which is equivalent to:

$$\min u \quad \text{s.t.} \quad ue - A^T y \geq 0$$
$$e^T y = 1$$
$$y \geq 0$$
Rowboy’s Problem in Block-Matrix Form

\[
\begin{align*}
\min & \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \begin{bmatrix} y \\ u \end{bmatrix} \\
\begin{bmatrix} -A^T & e \\ e^T & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} & \geq \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
\end{align*}
\]

\[y \geq 0\]

\[u \text{ free}\]

Note: Rowboy’s problem is dual to colgirl’s.
MiniMax Theorem

Let $x^*$ denote colgirl’s solution to her max–min problem. Let $y^*$ denote rowboy’s solution to his min–max problem. Then

$$\max_x y^T Ax = \min_y y^T Ax^*.$$

Proof.
From *Strong Duality Theorem*, we have

$$u^* = v^*.$$

Also,

$$v^* = \min_i e_i^T Ax^* = \min_y y^T Ax^*$$

$$u^* = \max_j y^T Ae_j = \max_x y^T Ax$$

QED
set ROWS;
set COLS;
param A {ROWS,COLS} default 0;

var x{COLS} >= 0;
var v;

maximize zot: v;

subject to ineqs {i in ROWS}:
    sum{j in COLS} -A[i,j] * x[j] + v <= 0;

subject to equal:
    sum{j in COLS} x[j] = 1;
AMPL Data

data;
set ROWS := P S R;
set COLS := P S R;
param A: P S R :=
    P 0 1 -2
    S -3 0 4
    R 5 -6 0
;

solve;
printf {j in COLS}: "  %3s %10.7f 
", j, 102*x[j];
printf {i in ROWS}: "  %3s %10.7f 
", i, 102*ineqs[i];
printf: "Value = %10.7f 
", 102*v;
AMPL Output

ampl gamethy.mod
LOQO: optimal solution (12 iterations)
primal objective \(-0.1568627451\)
  dual objective \(-0.1568627451\)
  P 40.0000000
  S 36.0000000
  R 26.0000000
  P 62.0000000
  S 27.0000000
  R 13.0000000
Value = \(-16.0000000\)
Dual of Problems in General Form

Consider:

\[
\begin{align*}
\max c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

Rewrite equality constraints as pairs of inequalities:

\[
\begin{align*}
\max c^T x \\
Ax &\leq b \\
-Ax &\leq -b \\
x &\geq 0
\end{align*}
\]
Put into block-matrix form:

\[
\begin{align*}
\max & \quad c^T x \\
\begin{bmatrix} A & -A \end{bmatrix} x & \leq \begin{bmatrix} b \\ -b \end{bmatrix} \\
x & \geq 0
\end{align*}
\]

Dual is:

\[
\begin{align*}
\min & \quad \begin{bmatrix} b & -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} \\
\begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} & \geq c \\
y^+, y^- & \geq 0
\end{align*}
\]
Which is equivalent to:

\[
\begin{align*}
\min b^T(y^+ - y^-) \\
A^T(y^+ - y^-) &\geq c \\
y^+, y^- &\geq 0
\end{align*}
\]

Finally, letting \( y = y^+ - y^- \), we get

\[
\begin{align*}
\min b^T y \\
A^T y &\geq c \\
y &\text{free.}
\end{align*}
\]

**Moral:**

- Equality constraints \(\implies\) free variables in dual.
- Inequality constraints \(\implies\) nonnegative variables in dual.

**Corollary:**

- Free variables \(\implies\) equality constraints in dual.
- Nonnegative variables \(\implies\) inequality constraints in dual.
A Real-World Example

The Ultra-Conservative Investor

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
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<tbody>
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<td>0.851</td>
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<td>0.735</td>
<td>0.716</td>
<td>0.662</td>
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<td>0.768</td>
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<td>1.318</td>
<td>1.123</td>
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<tr>
<td>1976</td>
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<td>1.236</td>
<td>1.266</td>
<td>1.280</td>
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<td>1.025</td>
<td>0.960</td>
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<td>1.002</td>
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<td>0.974</td>
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<td>1.030</td>
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<td>1.200</td>
</tr>
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<td>1.064</td>
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<td>1.146</td>
<td>1.012</td>
<td>1.326</td>
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<td>0.978</td>
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<td>1.307</td>
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<td>1.048</td>
<td>2.212</td>
</tr>
<tr>
<td>1980</td>
<td>1.127</td>
<td>0.947</td>
<td>1.323</td>
<td>1.337</td>
<td>1.367</td>
<td>1.031</td>
<td>1.226</td>
<td>1.296</td>
</tr>
<tr>
<td>1981</td>
<td>1.156</td>
<td>1.003</td>
<td>0.949</td>
<td>0.963</td>
<td>0.990</td>
<td>1.073</td>
<td>0.977</td>
<td>0.688</td>
</tr>
<tr>
<td>1982</td>
<td>1.117</td>
<td>1.465</td>
<td>1.215</td>
<td>1.187</td>
<td>1.213</td>
<td>1.311</td>
<td>0.981</td>
<td>1.084</td>
</tr>
<tr>
<td>1983</td>
<td>1.092</td>
<td>0.985</td>
<td>1.224</td>
<td>1.235</td>
<td>1.217</td>
<td>1.080</td>
<td>1.237</td>
<td>0.872</td>
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<td>1.186</td>
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<td>1.086</td>
<td>1.156</td>
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<tr>
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<td>1.052</td>
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<td>0.959</td>
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<td>1.244</td>
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<tr>
<td>1988</td>
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<td>1.086</td>
<td>1.165</td>
<td>1.179</td>
<td>1.165</td>
<td>1.076</td>
<td>1.283</td>
<td>0.861</td>
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<tr>
<td>1989</td>
<td>1.087</td>
<td>1.212</td>
<td>1.316</td>
<td>1.292</td>
<td>1.204</td>
<td>1.142</td>
<td>1.105</td>
<td>0.977</td>
</tr>
<tr>
<td>1990</td>
<td>1.080</td>
<td>1.054</td>
<td>0.968</td>
<td>0.938</td>
<td>0.830</td>
<td>1.083</td>
<td>0.766</td>
<td>0.922</td>
</tr>
<tr>
<td>1991</td>
<td>1.057</td>
<td>1.193</td>
<td>1.304</td>
<td>1.342</td>
<td>1.594</td>
<td>1.161</td>
<td>1.121</td>
<td>0.958</td>
</tr>
<tr>
<td>1992</td>
<td>1.036</td>
<td>1.079</td>
<td>1.076</td>
<td>1.090</td>
<td>1.174</td>
<td>1.076</td>
<td>0.878</td>
<td>0.926</td>
</tr>
<tr>
<td>1993</td>
<td>1.031</td>
<td>1.217</td>
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<td>1.110</td>
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<td>1.146</td>
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<tr>
<td>1994</td>
<td>1.045</td>
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<td>1.012</td>
<td>0.999</td>
<td>0.968</td>
<td>0.965</td>
<td>1.078</td>
<td>0.990</td>
</tr>
</tbody>
</table>

Consider again the historical return on investment data: We can view this as a payoff matrix in a game between Fate and the Investor.
Fate’s Conspiracy

The columns represent pure strategies for our conservative investor. The rows represent how history might repeat itself. Of course, for next year (1995), Fate won’t just repeat a previous year but, rather, will present some mixture of these previous years. Likewise, the investor won’t put all of her money into one asset. Instead she will put a certain fraction into each. Using this data in the game-theory AMPL model, we get the following mixed-strategy percentages for Fate and for the investor.

Investor’s Optimal Asset Mix:  

<table>
<thead>
<tr>
<th>Asset Class</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>US 3-MONTH T-BILLS</td>
<td>93.9</td>
</tr>
<tr>
<td>NASDAQ COMPOSITE</td>
<td>5.0</td>
</tr>
<tr>
<td>EAFE</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Mean, old Fate’s Mix:  

<table>
<thead>
<tr>
<th>Year</th>
<th>Fate Mix</th>
</tr>
</thead>
<tbody>
<tr>
<td>1992</td>
<td>28.1</td>
</tr>
<tr>
<td>1993</td>
<td>7.8</td>
</tr>
<tr>
<td>1994</td>
<td>64.1</td>
</tr>
</tbody>
</table>

The value of the game is the investor’s expected return: 4.10%.