2. Union - Find Algorithms
An illustration of the importance of algorithm design

Sedgewick, Algorithms in C, Chapter 1

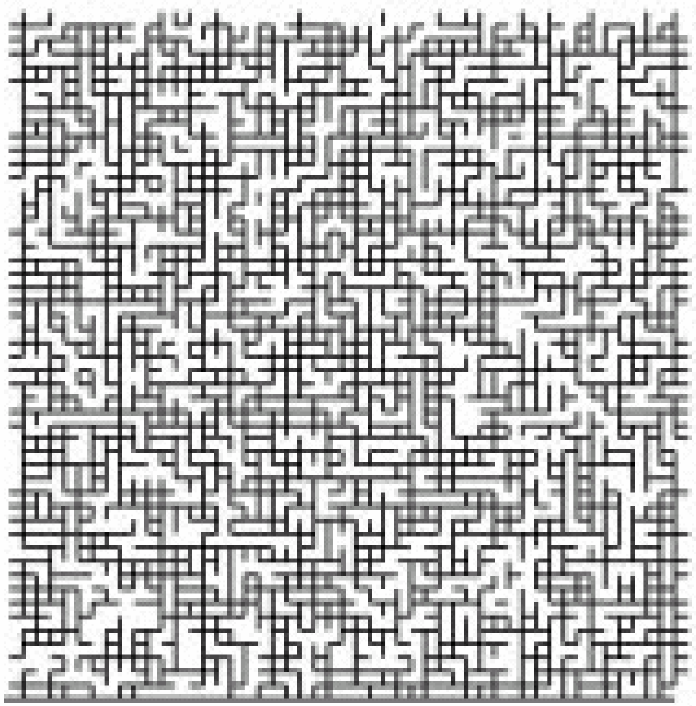
Slides from

http://www.cs.princeton.edu/courses/archive/spring03/cs226/lectures/intro.4up.pdf

An Example Problem: Network Connectivity

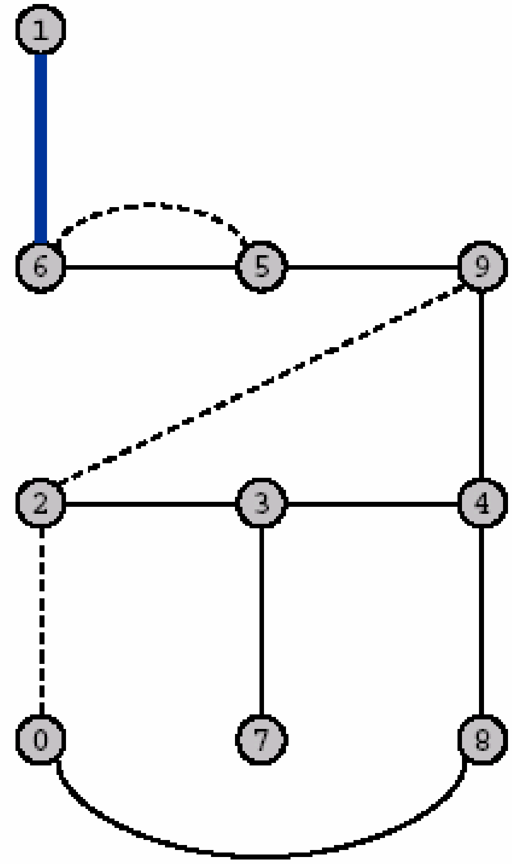
Network connectivity.

- Nodes at grid points.
- Add connections between pairs of nodes.
- Is there a path from node A to node B?



Network Connectivity

in	out	evidence
3 4	3 4	
4 9	4 9	
8 0	8 0	
2 3	2 3	
5 6	5 6	
2 9		(2-3-4-9)
5 9	5 9	
7 3	7 3	
4 8	4 8	
5 6		(5-6)
0 2		(2-3-4-8-0)
6 1	6 1	



Union-Find Abstraction

What are critical operations we need to support?

- N objects.
  - grid points
- FIND: test whether two objects are in same set.
  - is there a connection between A and B?
- UNION: merge two sets.
  - add a connection

Design efficient data structure to store connectivity information and algorithms for UNION and FIND.

- Number of objects and operations can be huge.

22

Another Application: Image Processing

Find connected components.

- Read in a 2D color image and find regions of connected pixels that have the same color.

Original

Labeled

23

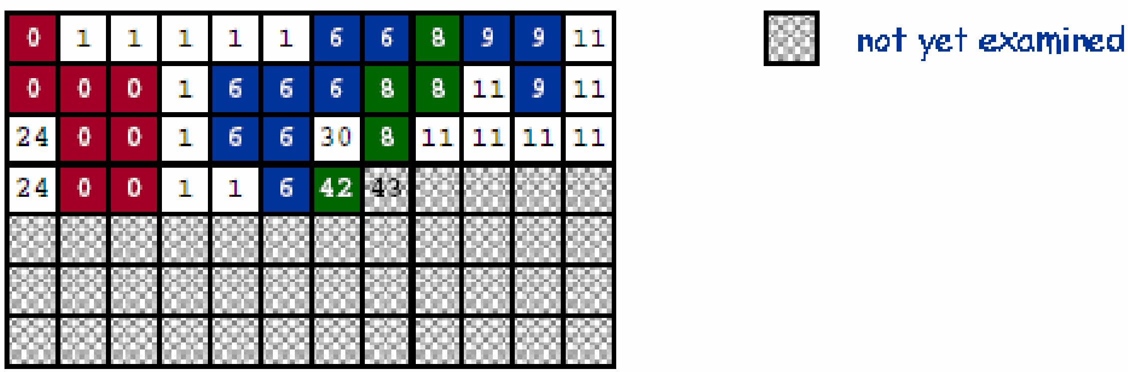
Another Application: Image Processing

Find connected components.

- Read in a 2D color image and find regions of connected pixels that have the same color.

One-pass algorithm.

- Initialize each pixel to be its own component.
- Examine pixels from left to right and top to bottom.
  - if a neighboring cell is the same color, merge current cell into same component



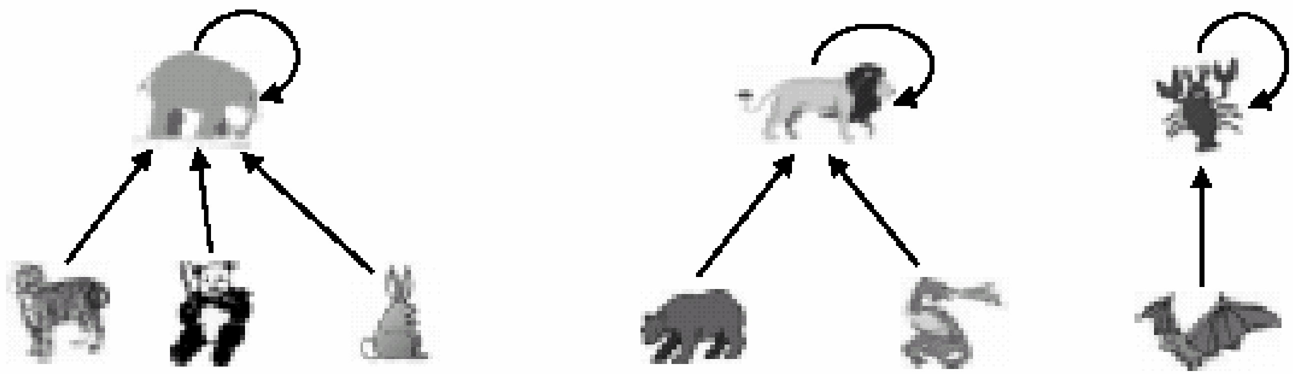
Objects

Elements are arbitrary objects in a network.

- Pixels in a digital photo.
- Computers in a network.
- Transistors in a computer chip.
- Web pages on the Internet.
- When programming, convenient to name them 0 to N-1.
- When drawing, fun to use animals!

26

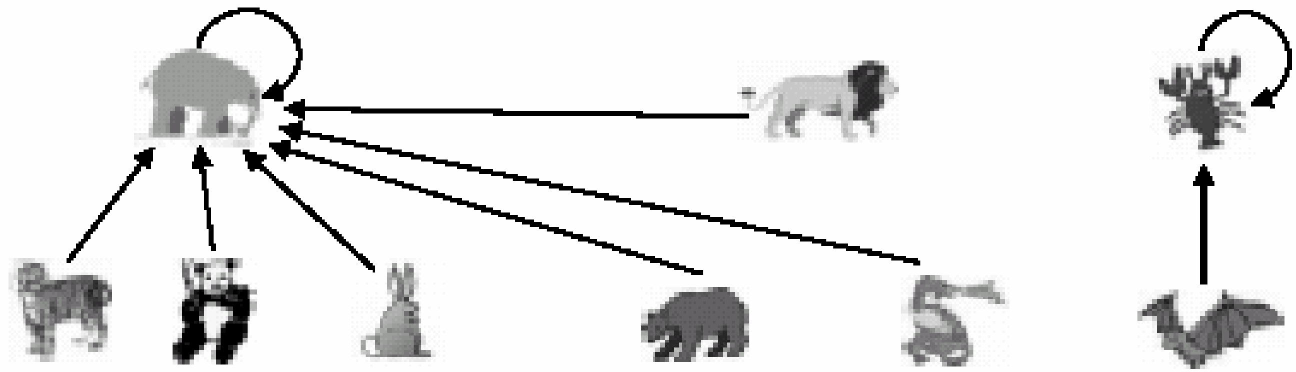
Quick-Find



id[tiger] = id[panda] = id[bunny] = id[elephant] = elephant
id[bear] = id[dragon] = id[lion] = lion
id[bat] = id[lobster] = lobster

27

Quick-Find



Union(tiger, bear)

29

Quick-Find Algorithm

Data structure.

- Maintain array `id[]` with name for each component.
- If p and q are connected, then same id.
- Initialize `id[i] = i`.

```
for (i = 0; i < N; i++)
   id[i] = i;
```

FIND. To check if p and q are connected, check if they have the same id.

```
if (id[p] == id[q])
   // already connected
```

UNION. To merge components containing p and q, change all entries with `id[p]` to `id[q]`.

```
pid = id[p];
for (i = 0; i < N; i++)
   if (id[i] == pid)
      id[i] = id[q];
```

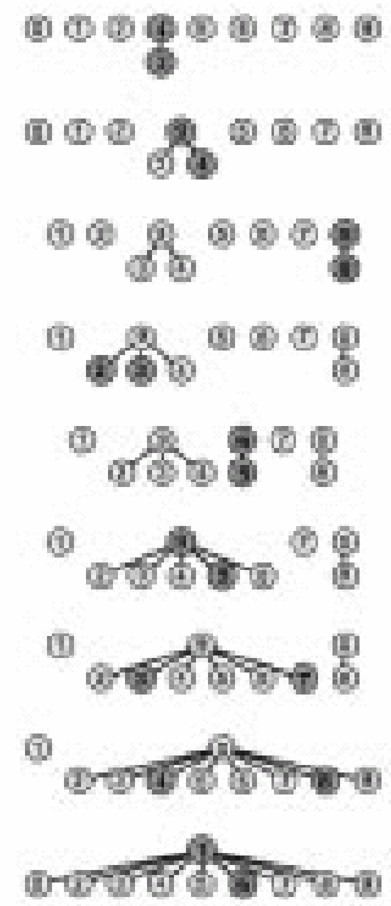
Analysis.

- FIND takes constant number of operations.
- UNION takes time proportional to N.

31

Quick-Find

3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	9	9	5	6	7	8	9
8-0	0	1	2	9	9	5	6	7	0	9
2-3	0	1	9	9	9	5	6	7	0	9
5-6	0	1	9	9	9	6	6	7	0	9
5-9	0	1	9	9	9	9	9	7	0	9
7-3	0	1	9	9	9	9	9	9	0	9
4-8	0	1	0	0	0	0	0	0	0	0
6-1	1	1	1	1	1	1	1	1	1	1



30

Problem Size and Computation Time

Rough standard for 2000.

- 10^9 operations per second.
- 10^9 words of main memory.
- Touch all words in approximately 1 second. (unchanged since 1950!)

Ex. Huge problem for quick find.

- 10^{10} edges connecting 10^9 nodes.
- Quick-find might take 10^{20} operations. (10 ops per query)
- 3,000 years of computer time!

Paradoxically, quadratic algorithms get worse with newer equipment.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

32

Quick-Union

Data structure: disjoint forests.

- Maintain array `id[]` with name for each component.
- If p and q are connected, p and q have same root, where
  - root(p) = `id[id[id[...id[p]...]]]`
  - go until it doesn't change

FIND. Check if p and q have same root.

```
for (i = p; i != id[i]; i = id[i]) ;
for (j = q; j != id[j]; j = id[j]) ;
if (i == j) // connected
```

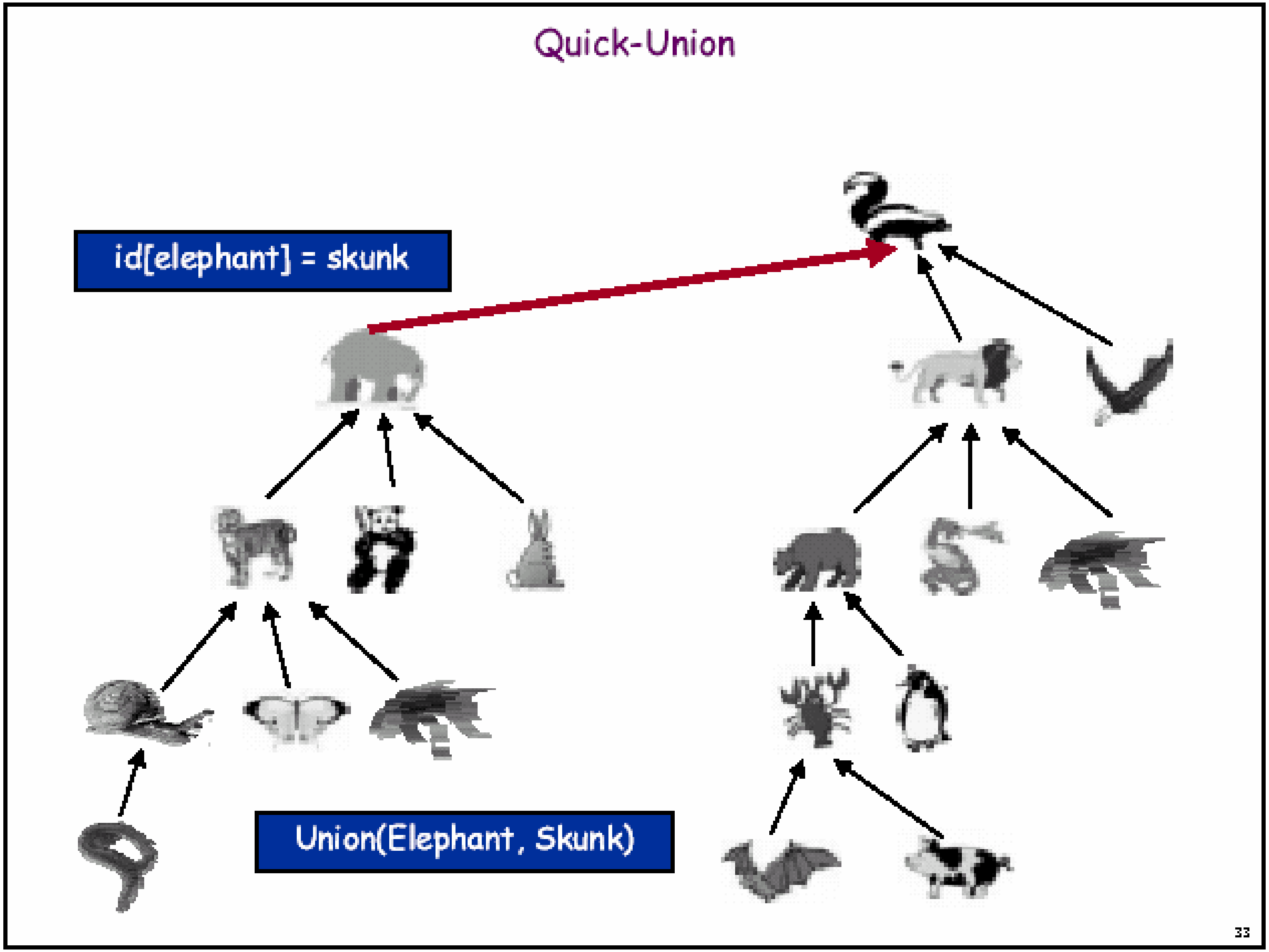
UNION. Set the id of p's root to q's root.

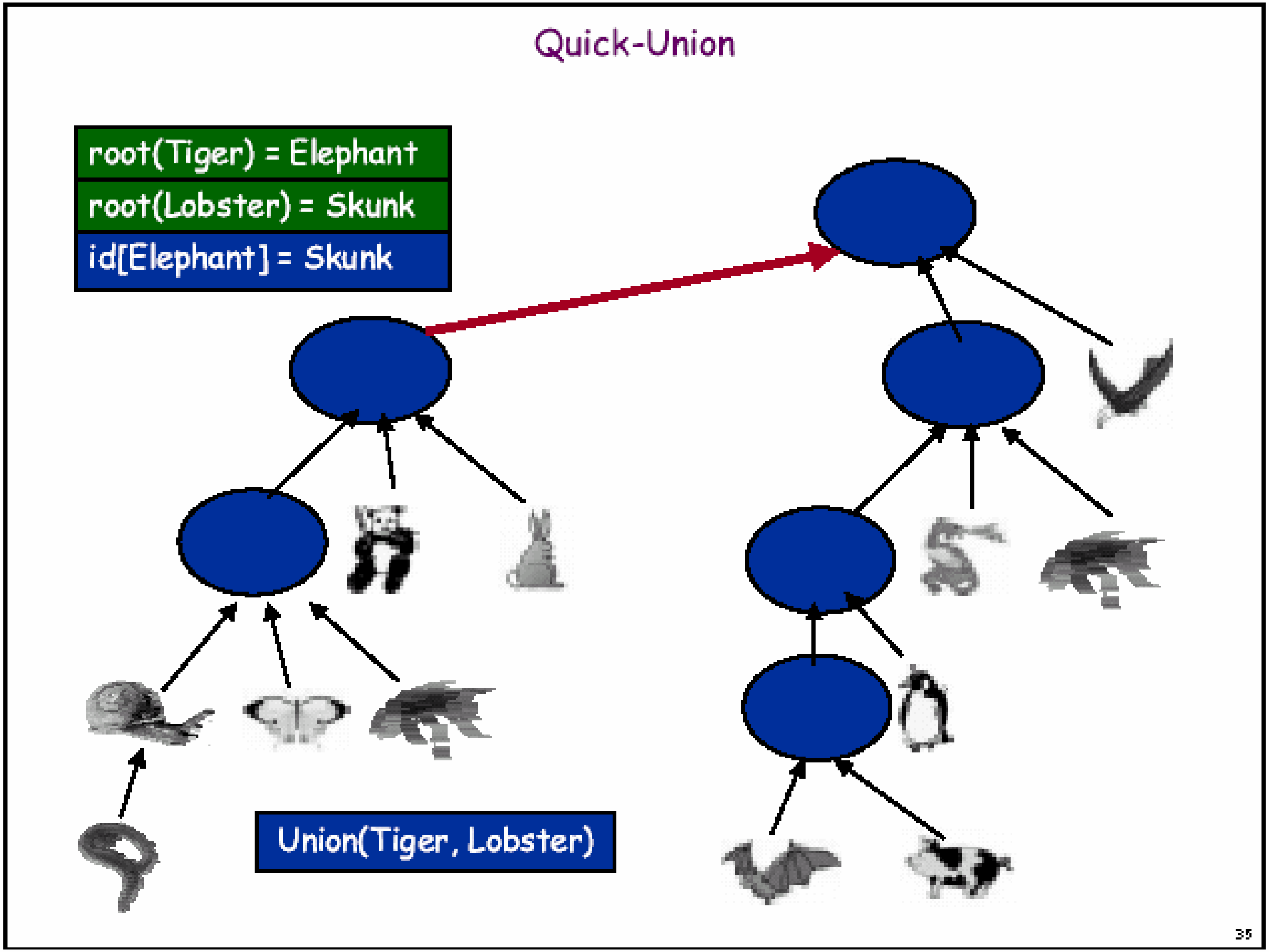
```
id[i] = j;
```

Analysis.

- FIND takes time proportional to depth of p and q in tree.
  - could be proportional to N
- UNION takes constant time, given roots.

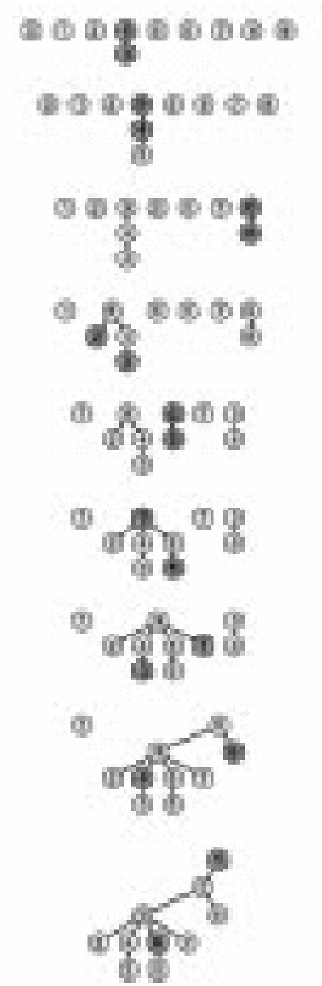
37


Quick-Union
id[elephant] = skunk
Union(Elephant, Skunk)
33


Quick-Union
root(Tiger) = Elephant
root(Lobster) = Skunk
id[Elephant] = Skunk
Union(Tiger, Lobster)
35

Quick-Union

3-4	0	1	2	4	4	5	6	7	8	9
4-9	0	1	2	4	9	5	6	7	8	9
8-0	0	1	2	4	9	5	6	7	0	9
2-3	0	1	9	4	9	5	6	7	0	9
5-6	0	1	9	4	9	6	6	7	0	9
5-9	0	1	9	4	9	6	9	7	0	9
7-3	0	1	9	4	9	6	9	9	0	9
4-8	0	1	9	4	9	6	9	9	0	0
6-1	1	1	9	4	9	6	9	9	0	0



36

Weighted Quick-Union

Quick-find defect.

- UNION too expensive.
- Trees are flat, but too hard to keep them flat.

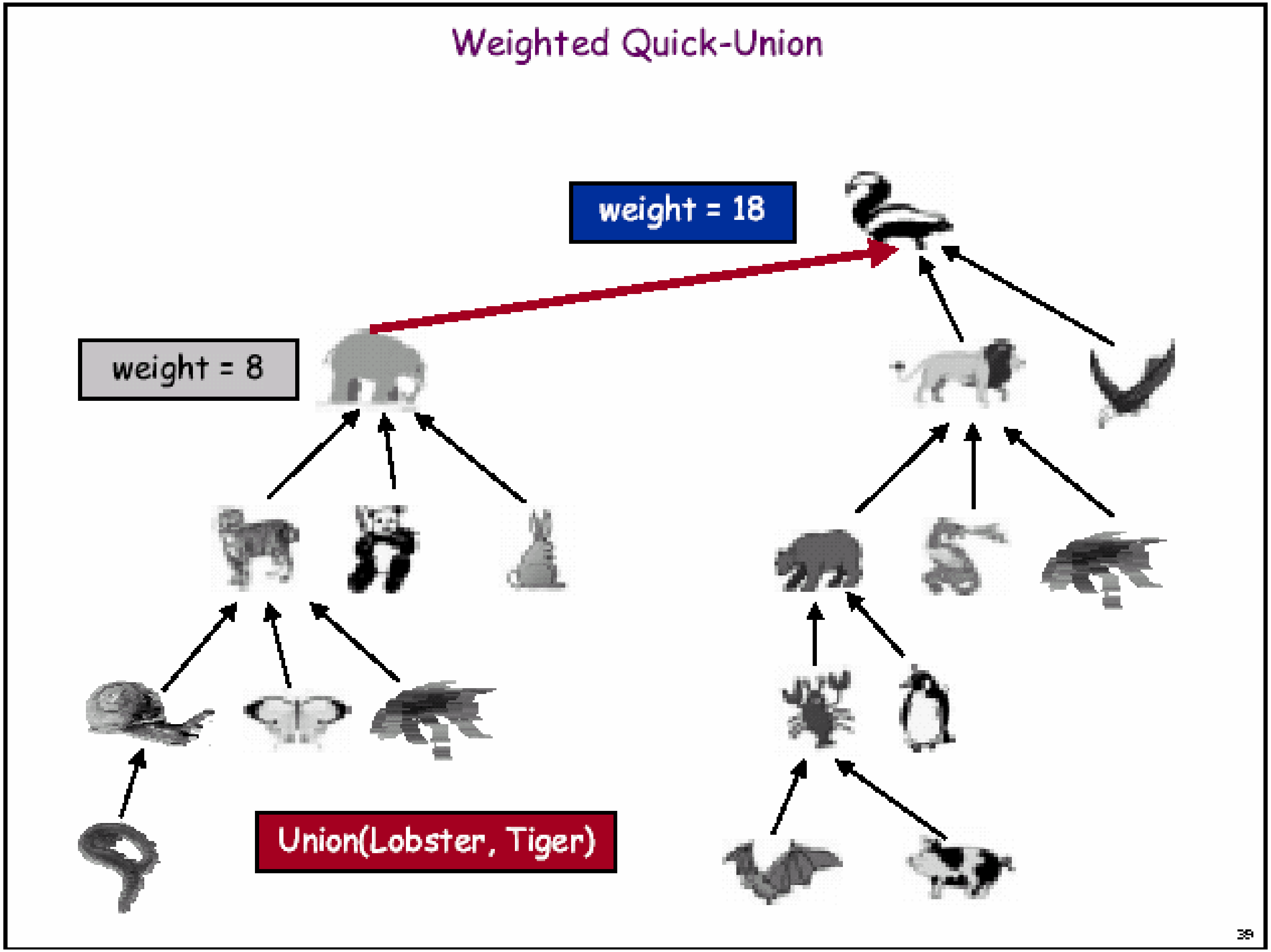
Quick-union defect.

- FIND could be too expensive.
- Trees could get tall.

Weighted quick-union.

- Modify quick-union to avoid tall trees.
- Keep track of size of each component.
- Balance by linking small tree below large one.

38


Weighted Quick-Union
weight = 18
weight = 8
Union(Lobster, Tiger)
39

Weighted Quick-Union

Data structure: disjoint forests.

- Also maintain array `wt[i]` that counts the number of nodes in the tree rooted at i.

FIND. Same as quick union.

UNION. Same as quick union, but:

- Merge smaller tree into the larger tree.
- Update the `wt[]` array.

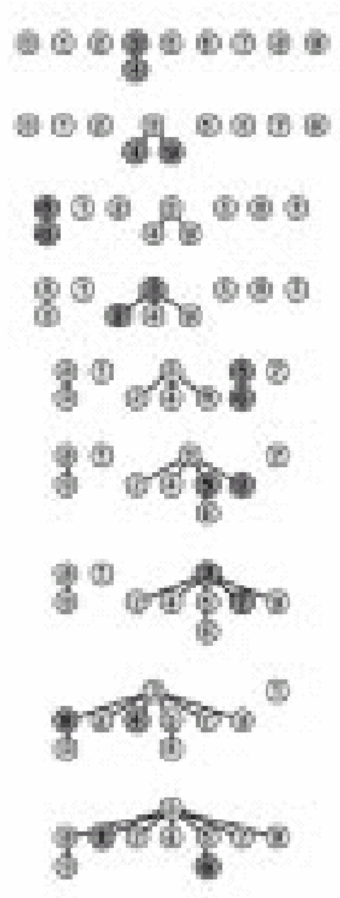
```
if (wt[i] < wt[j]) {
   id[i] = j;
   wt[j] += wt[i];
}
else {
   id[j] = i;
   wt[i] += wt[j];
}
```

Analysis.

- FIND takes time proportional to depth of p and q in tree.
  - depth is at most lg N
- UNION takes constant time, given roots.

Weighted Quick-Union

3-4	0	1	2	3	3	5	6	7	8	9
4-9	0	1	2	3	3	5	6	7	8	3
8-0	8	1	2	3	3	5	6	7	8	3
2-3	8	1	3	3	3	5	6	7	8	3
5-6	8	1	3	3	3	5	5	7	8	3
5-9	8	1	3	3	3	3	5	7	8	3
7-3	8	1	3	3	3	3	5	3	8	3
4-8	8	1	3	3	3	3	5	3	3	3
6-1	8	3	3	3	3	3	5	3	3	3



Weighted Quick-Union

Is performance improved?

- Theory: lg N per union or find operation.
- Practice: constant time.

Ex. Huge practical problem.

- 10^{10} edges connecting 10^{9} nodes.
- Reduces time from 3,000 years to 1 minute.
- Supercomputer wouldn't help much.
- Good algorithm makes solution possible.

Stop at guaranteed acceptable performance?

- Not hard to improve algorithm further.

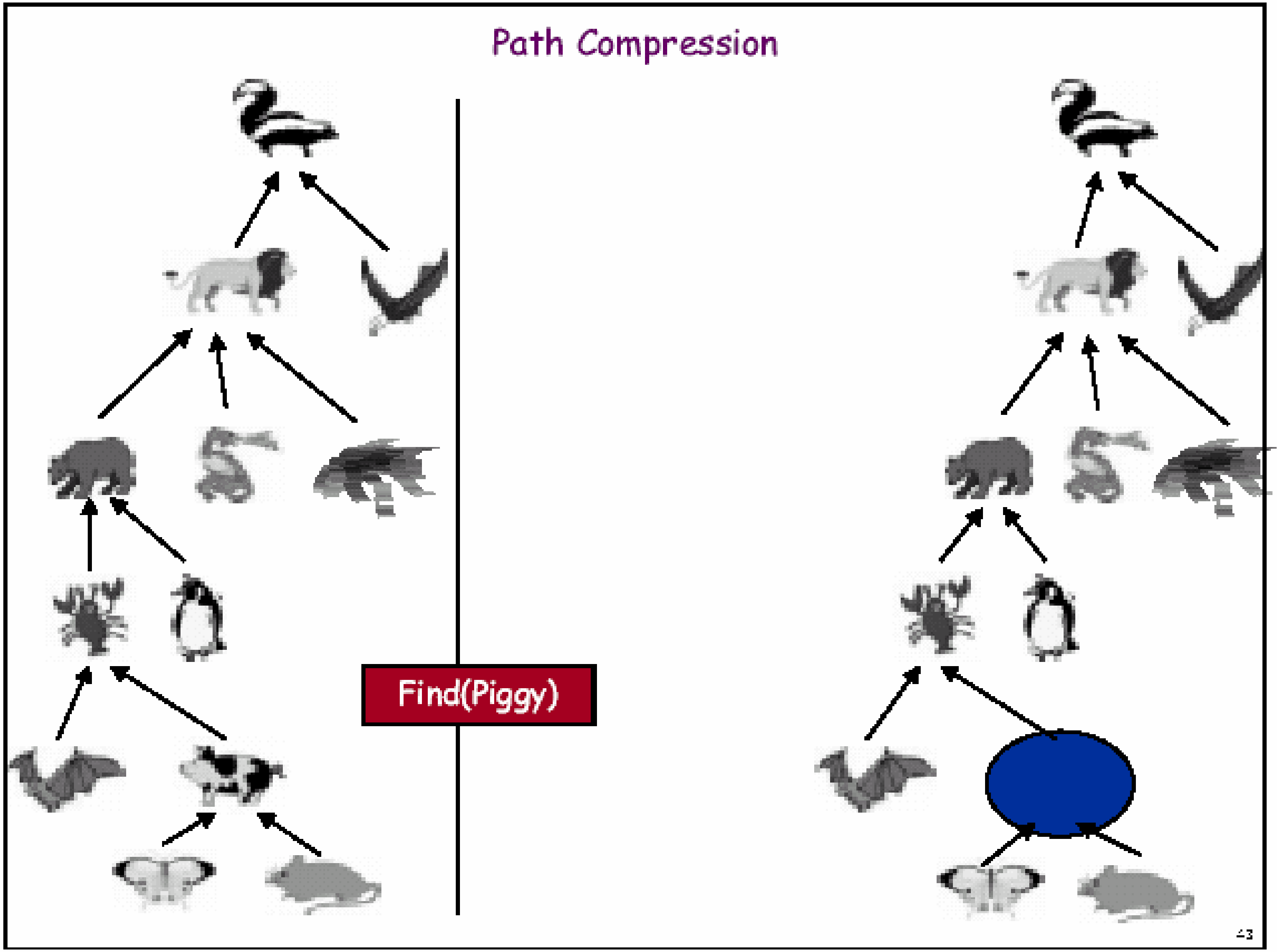
Weighted Quick-Union with Path Compression

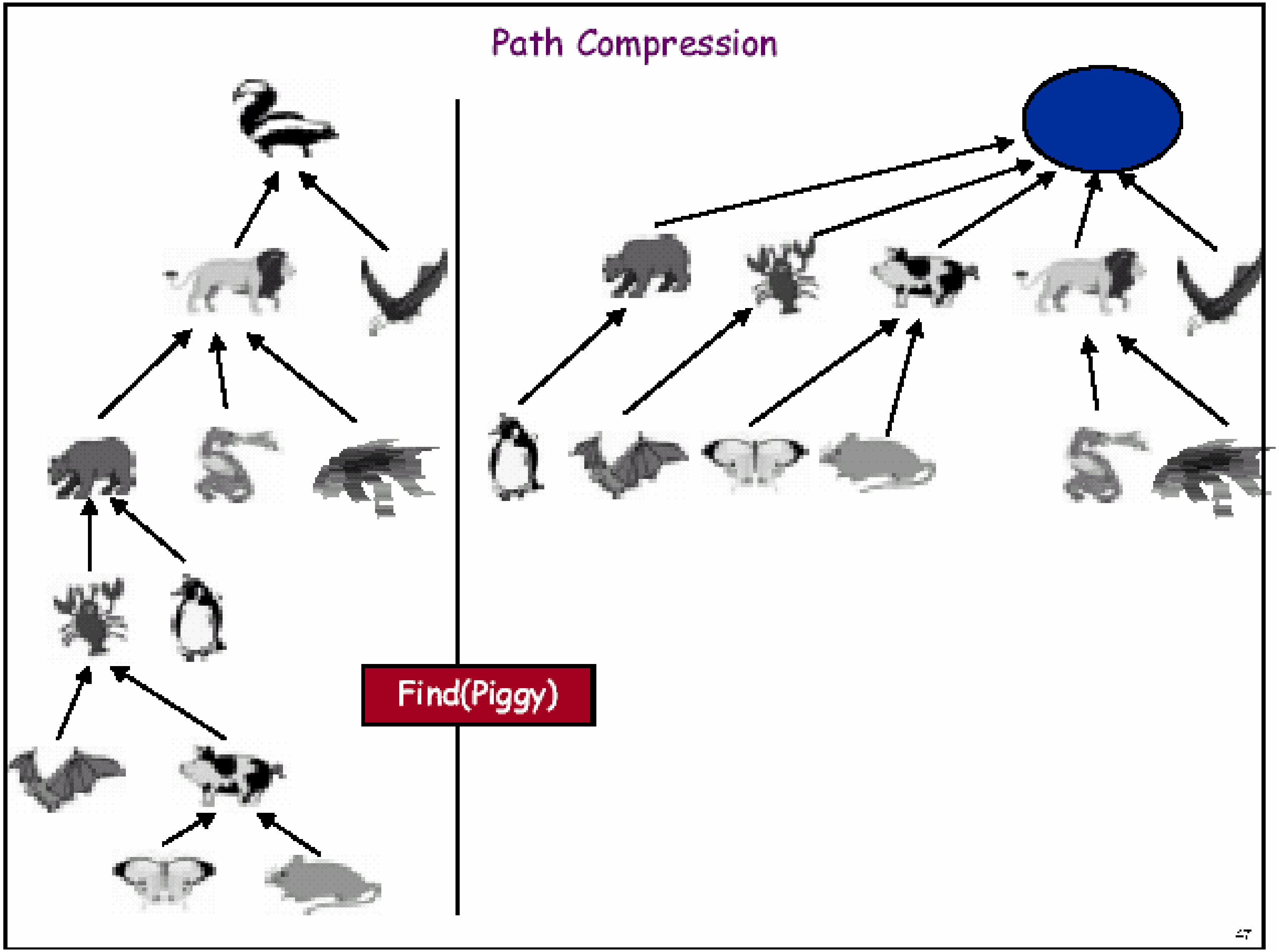
Path compression.

- Modify weighted quick-union to compress tree.
- Make second pass from p and q up to root, and set the id of every examined node to the new root.

```
for (i = p; i != id[i]; i = id[i])
   id[i] = root;
for (j = q; j != id[j]; j = id[j])
   id[j] = root;
```

- No reason not to!
- In practice, keeps tree almost completely flat.


Path Compression
Find(Piggy)


Path Compression
Find(Piggy)

Weighted Quick-Union with Path Compression

Theorem. A sequence of M union and find operations on N elements takes $O(N + M \lg^* N)$ time.

- Proof is difficult.
- But the algorithm is still simple!

Remark. $\lg^* N$ is a constant in this universe.

N	lg* N
2	1
4	2
16	3
65536	4
2^{65536}	5

Linear algorithm?

- Cost within constant factor of reading in the data.
- Theory: WQUPC is not quite linear.
- Practice: WQUPC is linear.

50

Lessons

Union-find summary.

- Online algorithm can solve problem while collecting data for "free."

Algorithm	Time
Quick-find	M N
Quick-union	M N
Weighted	N + M log N
Path compression	N + M log N
Weighted + path	5 (M + N)

"Trivial" algorithms can be useful.

- Start with simple algorithm.
  - don't use for large problems
  - can't use for huge problems
- Fast performance on test data OK.
- Strive for worst-case performance guarantees.
  - might be nontrivial to analyze
- Identify fundamental abstractions.
  - union-find
  - disjoint forests