# Inclusion-Exclusion Principle

In the set $S = \{1, 2, \ldots, 100\}$, one out of every six numbers is a multiple of 6, so that the total of multiples of 6 in $S$ is $[100/6] = [16.666\ldots] = 16$ (here $[x]$ is the integer part of $x$). Similarly, the total number of multiples of 7 in $S$ is $[100/7] = 14$.

How many numbers in $S$ are multiples of 6 or 7?

The answer is not $14 + 16 = 30$. The reason is that the sum $14 + 16$ counts twice those numbers that are both a multiple of 6 AND a multiple of 7; i.e., the multiples of 42. How many multiples of 42 were counted twice? As before, we compute $[100/42] = 2$. Then the correct answer is that there are $14 + 16 - 2 = 28$ numbers in $S$ that are multiples of 6 or 7.

How many numbers in $S$ are multiples of 2, 3, or 5?

In this case there are 50 multiples of 2, 33 multiples of 3 and 20 multiples of 5. But the answer is clearly not $50 + 33 + 20 = 103$. This sum counts twice the numbers that are multiples of 2 and 3 for instance. We must subtract 16 multiples of 6, 10 multiples of 10 and 6 multiples of 15.

It seems as if $50 + 33 + 20 - 16 - 10 - 6 = 71$ is the final answer, but it is not! The multiples of 30 were counted 3 times and eliminated 3 times. They are not accounted for. We have to add 3 multiples of 30 to get the correct answer: $50+33+20-16-10-6+3 = 74$.

The inclusion-exclusion principle tells us how to keep track of what to add and what to subtract in problems like the above:

Let $S$ be a finite set, and suppose there is a list of $r$ properties that every element of $S$ may or may not have. We call $S_1$ the subset of elements of $S$ that have property 1; $S_{1,2}$ the subset of elements in $S$ that have properties 1 and 2, etc.
Notice that $\bigcup S_i$ is the subset of elements of $S$ that have at least one of the $r$ properties. To count these elements, we

- add the number of elements that have at least one property:

$$|S_1| + |S_2| + \ldots + |S_r|$$

- subtract the number of elements that have at least two properties:

$$-|S_{1,2}| - |S_{1,3}| - \ldots - |S_{r-1,r}|$$

- add the number of elements that have at least three properties:

$$+|S_{1,2,3}| + |S_{1,2,4}| + \ldots + |S_{r-2,r-1,r}|$$

$$\vdots$$

- add/subtract the number of elements that have all the properties:

$$\pm|S_{1,2,\ldots,r}|$$

**Note:** In some problems you need to compute how many elements do not have ANY of the $r$ properties. These are precisely the elements that are not in $\bigcup S_i$. In this case you can compute the total of elements in $\bigcup S_i$ using the inclusion-exclusion principle, and subtract that from the number of elements in $S$. Problem 6 is an important example of this trick.

Problems ⟶

1. In a survey on the chewing gum preferences of baseball players, it was found that
   - 22 like fruit.
   - 25 like spearmint.
   - 39 like grape.
   - 9 like spearmint and fruit.
   - 17 like fruit and grape.
   - 20 like spearmint and grape.
   - 6 like all flavors.
   - 4 like none.

   How many players were surveyed?

2. What is the probability that a poker hand has no aces?

3. If 8 dies are rolled, what is the probability that all 6 numbers appear?

4. Recall that the totient function $\varphi(n)$ counts how many numbers less than $n$ are relatively prime with $n$.
   - Determine $\varphi(2^n)$.
   - Determine $\varphi(2^n p)$, where $p$ is an odd prime.
   - Determine $\varphi(p_1 p_2)$, where $p_1$ and $p_2$ are different primes.
   - Find a formula for $\varphi(n)$ if the prime decomposition of $n$ is $n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$.

5. Recall that the Möbius function $\mu(n)$ is 0 if $n$ contains a square factor and is $(-1)^r$ if $n$ is the product of $r$ different primes. For any $n \geq 2$, prove that $\sum_{d|n} \mu(d) = 0$.

6. A *derangement* of $(1, 2, \dots, n)$ is a permutation that moves every number away from its correct position. For example $(2, 5, 4, 1, 3)$ is a derangement, but $(2, 5, 3, 1, 4)$ is not. How many derangements of $(1, 2, \dots, n)$ are there?