11.4 Maximizing and minimizing functions of two variables

Horizontal tangent plane so solve system of equations to locate the critical points.

Recall in the calculus of one variable, if $y = f(x)$ is defined on a set S, then there is a **relative maximum value** at x0 if $f(x0) \geq f(x)$ for all x in S near x0, and there is a **relative minimum value** at x0 if $f(x0) \leq f(x)$ for all x in S near x0. A relative maximum locates the top of a hill, while a relative minimum locates the bottom of a valley.

Example. The function $f(x) = \sin(x)$ has a relative maximum value at $x = \pi/2 = 1.571$, at $x = 5\pi/2 = 7.854$, and at $x = -3\pi/2 = -4.712$. It has a relative minimum value at $x = -\pi/2 = -1.571$, at $x = 3\pi/2 = 4.712$, and $x = 7\pi/2 = 10.996$.

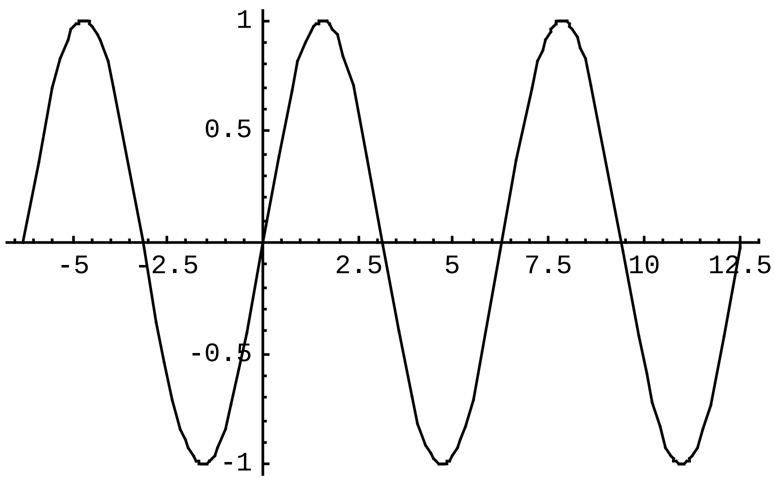

Figure 11.3.1. The function $f(x) = \sin(x)$

In the same way a function of two variables has a relative maximum at the top of a hill, while it has a relative minimum at the bottom of a valley. For example, the function $f(x,y) = 1 - x^2 - y^2 + 2x + 4y$ has the graph shown in Figure 11.3.2. There is a relative maximum at (1,2), ie where x = 1 and y = 2. This corresponds to the top of the hill shown by the graph, although the exact coordinates are hard to tell from the picture.

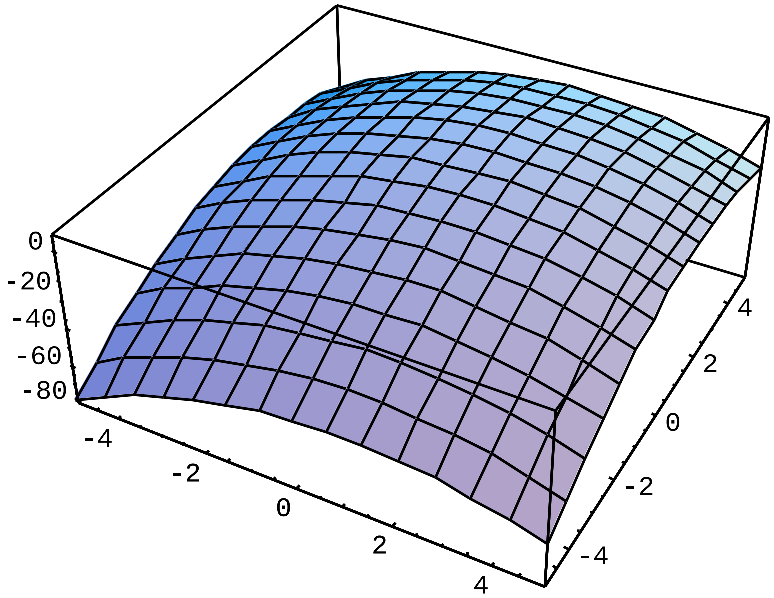

Figure 11.3.2. The function $f(x,y) = 1 - x^2 - y^2 + 2x + 4y$ has a relative maximum. The x-axis is the more nearly horizontal, while the y-axis seems to recede into the paper.

Similarly, figure 11.3.3 shows a relative minimum.

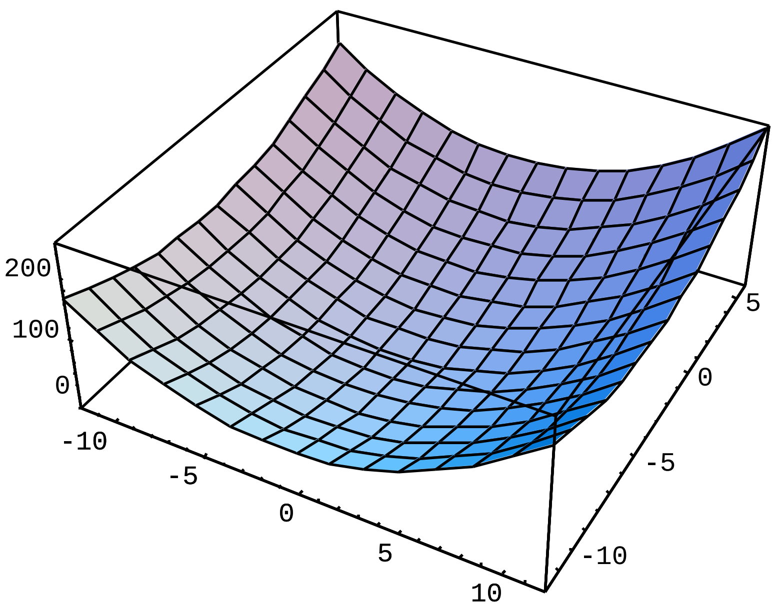

Figure 11.3.3. The function $f(x,y) = x^2 + y^2 + 2x + 8y$ has a relative minimum (at the bottom of the valley).

It is often important to locate the relative maximum or the relative minimum of a function, just as for a function of 1 variable it is common to seek the relative maximum or relative minimum. For one variable, if $y = f(x)$, we found candidates for a relative maximum or a relative minimum by solving the equation

$$f'(x) = 0.$$

This worked because at a maximum or minimum we expect the tangent line to be horizontal.

In the same way if $z = f(x,y)$, we expect the tangent plane to be horizontal at a relative maximum or a relative minimum. To be horizontal at (x_0, y_0), the tangent plane which has the form

$$z = z_0 + \partial f/\partial x\,(x_0, y_0)\,(x - x_0) + \partial f/\partial y(x_0, y_0)\,(y - y_0)$$

must also have the form

$$z = z_0$$

since a horizontal plane has constant z value. Thus we expect that at a relative maximum or a relative minimum we have both

$$\partial f/\partial x\,(x_0, y_0) = 0$$

and

$$\partial f/\partial y\,(x_0, y_0) = 0.$$

Definition. A **critical point** for $z = f(x,y)$ is a point (x_0, y_0) such that both

$$\partial f/\partial x\,(x_0, y_0) = 0$$

and

$$\partial f/\partial y\,(x_0, y_0) = 0.$$

Our discussion above says that

A relative maximum or a relative minimum occurs at a critical point.

Example. Find the critical points for $f(x,y) = 1 - x^2 - y^2 + 2x + 4y$.
Solution. We need $\partial f/\partial x = 0$ and $\partial f/\partial y = 0$. Hence

$$\partial f/\partial x = -2x + 2 = 0$$
$$\partial f/\partial y = -2y + 4 = 0$$

Hence $x = 1$ and $y = 2$. The critical point is $(1,2)$. In Figure 11.3.2, this tells the location of the top of the hill. This calculation locates the critical point much more easily than reading it from the graph.

Example. Find all critical points if

$$f(x,y) = x^2 - 2xy - y^2 + 10x - 6y$$

Solution.

We solve

$$\partial f/\partial x = 2x - 2y + 10 = 0$$
$$\partial f/\partial y = -2x - 2y - 6 = 0$$

This time the equations must be solved simultaneously. If we add them together, we find

$$-4y + 4 = 0,$$

so $y = 1$.
Now from the first equation we obtain

$$2x + 8 = 0,$$

so $x = -4$.
The critical point is $(-4,1)$.

Once we have found a critical point, a natural question to ask is what kind of critical point it is. We have seen above that it may be a relative maximum or a relative minimum. Another possibility is that it is neither a relative maximum nor a relative minimum. A common occurrence is a **saddle point,** named because it looks like the saddle on a horse. Figure 11.3.4 shows a saddle point. The saddle point is the point $(0,0)$ with a horizontal tangent line, exactly where you would sit if you sat on the saddle.

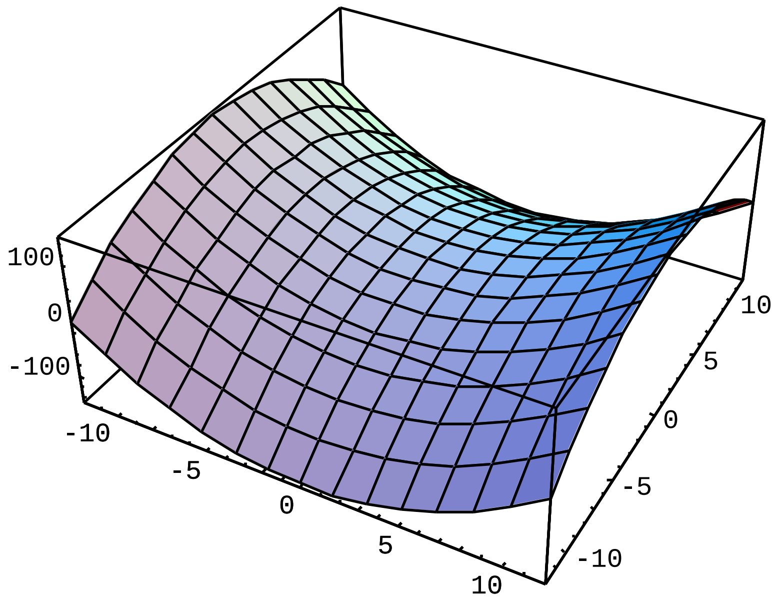

Figure 11.3.4. The graph of $f(x,y) = x^2 - y^2$. There is a saddle point at (0,0).

Here is the algebraic test to tell what kind of critical point one has--whether it is a relative maximum, a relative minimum, or a saddle point. It makes use of the second partial derivatives. This analog of the Second Derivative Test for functions of one variable is the most common method utilized to identify whether a critical point is a relative maximum or a relative minimum.

Theorem. **(Second Partials Test)** Suppose $z = f(x,y)$ has a critical point at (x_0, y_0), so $\partial f/\partial x(x_0,y_0) = 0$ and $\partial f/\partial y(x_0,y_0) = 0$.
Let

$$D = \partial^2 f/\partial x^2(x_0,y_0) \quad \partial^2 f/\partial y^2(x0,y0) - [\partial^2 f/\partial x\partial y(x0,y0)]^2$$

(1) If $D > 0$ and $\partial^2 f/\partial x^2(x_0,y_0) > 0$, then $f(x_0,y_0)$ is a relative minimum value.
(2) If $D > 0$ and $\partial^2 f/\partial x^2(x_0,y_0) < 0$, then $f(x_0,y_0)$ is a relative maximum value.
(3) If $D < 0$ then $f(x_0,y_0)$ is a saddle point (neither local max nor local min)
(4) In all other cases, you can conclude nothing.

Example. Find all the critical points of

$$f(x,y) = 3x^2 - 12 x + 2 y^2 + 16 y - 10$$

and tell what kind each is.
Solution. $\partial f/\partial x = 6x - 12 = 0$

$$\partial f/\partial y = 4y + 16 = 0$$

Hence

$$x = 2, y = -4.$$

There is only one critical point, at (2, -4).

Next we use the second partials test to identify the type of critical point this is.

$\partial^2 f/\partial x^2(x,y) = 6$
$\partial^2 f/\partial y^2(x,y) = 4$
$\partial^2 f/\partial x\partial y(x,y) = 0$

Hence $D = (6)(4) - (0)^2 = 24 > 0$.
Since $\partial^2 f/\partial x^2(x,y) = 6 > 0$, we conclude that there is a relative minimum at (2,-4).

Example. Locate and classify all critical points if

$$f(x,y) = x^2 - 2xy - y^2 - 12x + 16y$$

Solution.

First we locate the critical points:

$\partial f/\partial x = 2x - 2y - 12 = 0$

$\partial f/\partial y = -2x - 2y + 16 = 0$

From the first equation we see

$y = x - 6$

Substituting this into the second equation we see

$-2x - 2(x-6) + 16 = 0$

$-4x + 28 = 0$

$x = 7$

Hence $y = 7 - 6 = 1$.
The critical point is (7,1).

Next we use the Second Partials Test:

$\partial^2 f/\partial x^2(x,y) = 2$

$\partial^2 f/\partial y^2(x,y) = -2$

$\partial^2 f/\partial x\partial y(x,y) = -2$

Hence $D = (2)(-2) - (-2)^2 = -8$.
Hence (7,1) is a saddle point.

Sometimes there may be more than one critical point:

Example. Locate and classify all critical points if

$$f(x,y) = xy - x^3 - 2y^2$$

Solution.

We must have $\partial f/\partial x = 0$, or $y - 3x^2 = 0$
and $\partial f/\partial y = 0$, or $x - 4y = 0$.

Hence $x = 4y$ and $y = 3x^2$.
It follows $y = 3x^2 = 3(4y)^2 = 48y^2$
Hence $48y^2 - y = 0$

$y(48y - 1) = 0$

$y = 0$ or $y = 1/48$.

If $y = 0$, then $x = 4y = 0$.
If $y = 1/48$, then $x = 4y = 1/12$. Hence the critical points are (0,0) and (1/12, 1/48).

We must see which kind of critical point each one is.
Note $\partial^2 f/\partial x^2 = -6x$

$\partial^2 f/\partial y\partial x = 1$

$\partial^2 f/\partial y^2 = -4$.

Hence $D = (-6x)(-4) - (1)^2 = 24x - 1$.
At (0,0), $D = -1$ so (0,0) is a saddle point.
At (1/12, 1/48), $D = 24(1/12) - 1 = 1 > 0$, and moreover $\partial^2 f/\partial x^2 = -6x = -6(1/12) = -1/2$.
Hence (1/12, 1/48) is a relative maximum.

The value of finding maxima and minima for most scientists is in solving problems that actually arise in their work.

Example. A rectangular box must have volume 500 in^3. Find the shape that has the smallest mailing length (the sum of the three edge lengths).

Solution. Let x, y, z be the edge lengths. The mailing length L is the sum of the three edge lengths, so

$$L = x + y + z.$$

But the volume = x y z = 500, so
we may solve for z = 500 / (xy).
Thus $L = x + y + 500 / (xy) = x + y + 500\, x^{-1} y^{-1}$.
We want to minimize L, which is a function of 2 variables.

To do this, we solve $\partial L/\partial x = 0$ and $\partial L/\partial y = 0$.

$\partial L/\partial x = 1 - 500\, x^{-2} y^{-1} = 0$
$\partial L/\partial y = 1 - 500\, x^{-1} y^{-2} = 0$

We solve these two equations:

$$1 - \frac{500}{x^2 y} = 0$$

$$1 - \frac{500}{x y^2} = 0$$

Hence

$$x^2 y - 500 = 0$$
$$x y^2 - 500 = 0.$$

From the first equation

$$y = 500 / x^2.$$

Substituting into the second equation we have

$$x (500/x^2)^2 - 500 = 0$$

$$(500)(500)/ x^3 - 500 = 0$$
$$x^3 = 500$$
$$x = 500^{(1/3)}$$
$$y = 500 / x^2 = 500^{(1/3)}$$
$$z = 500/(xy) = 500^{(1/3)}$$

The answer is that the best box has $x = y = z = 500^{(1/3)} = 7.937$ inches.

Why does the Second Partials Test work?

The explanation is quite subtle and beyond the possibilities for this course. A portion of the test is easy to understand, however. If there is to be a relative maximum at (x,y) we check (among other things) whether

$\partial^2 f/\partial x^2$ (x,y) < 0.

This is the same as the Second Derivative Test for the function where y is kept fixed and only x changes. When $\partial^2 f/\partial x^2$ (x,y) < 0, the one-variable Second Derivative Test suggests that there is a relative maximum at (x,y). Hence if $\partial^2 f/\partial x^2$ (x,y) < 0 we expect the possibility of a relative maximum at (x,y).

Similarly if $\partial^2 f/\partial x^2$ (x,y) > 0, the one-variable Second Derivative Test suggests that there is a relative minimum at (x,y). Hence if $\partial^2 f/\partial x^2$ (x,y) > 0 we expect the possibility of a relative minimum at (x,y).

What is hard to understand is the role of D. Note that D includes information about $\partial^2 f/\partial x \partial y$ (x,y). This added information turns out to be essential in general. Even if $\partial^2 f/\partial x^2$ (x,y) < 0 it is possible that there is not a relative maximum at (x,y). This possibility is identifed by having D < 0, so there is a saddle point.