# CS3000: Algorithms & Data
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Lecture 18:

- Greedy Algorithms: Proof Techniques

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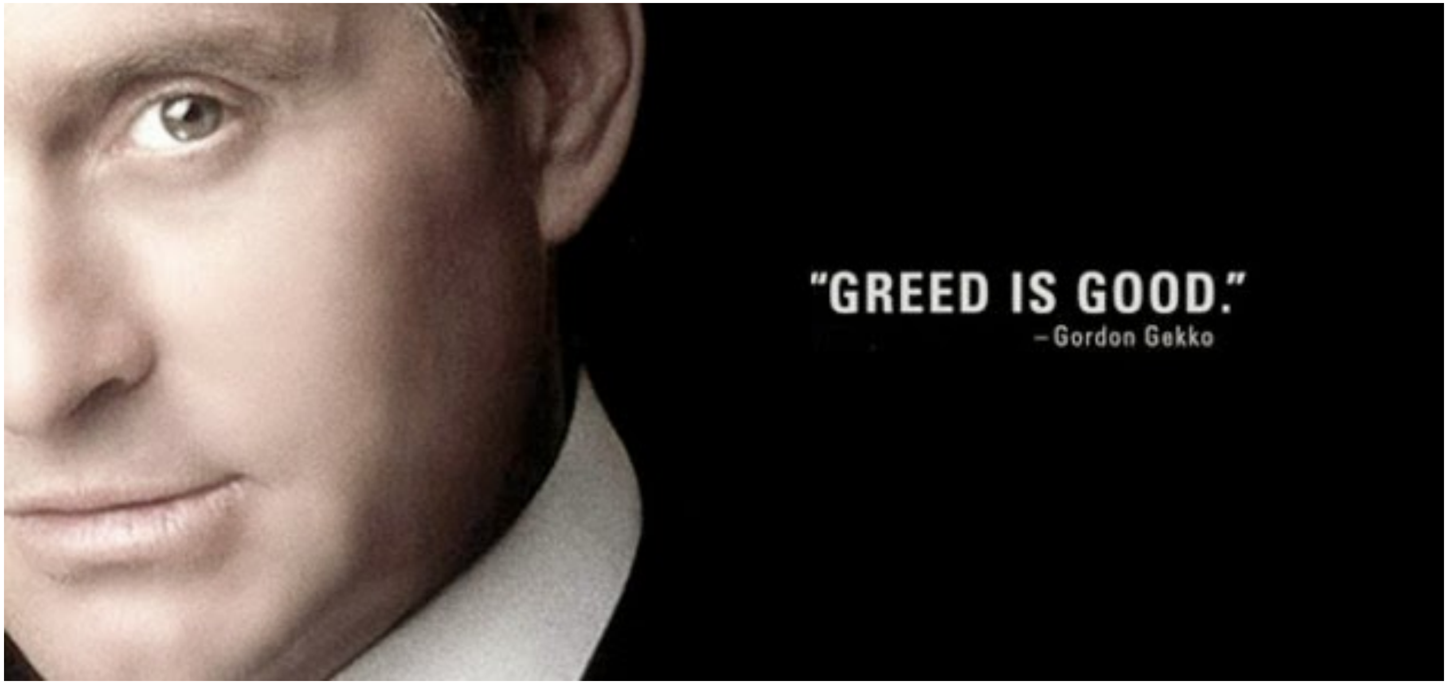
# Obligatory *Wall Street* Quotation




The movie *Wall Street*, however, is not.

# Greedy Algorithms

- What’s a greedy algorithm?
  - I know it when I see it
  - Roughly, an algorithm that builds a solution myopically and never looks back (compare to DP)
  - Typically, make a single pass over the input (e.g. Kruskal)

- Why care about greedy algorithms?
  - Greedy algorithms are the fastest and simplest algorithms imaginable, and sometimes they work!
  - Sometimes make useful heuristics when they don’t
  - Simplicity makes them easy to adapt to different models

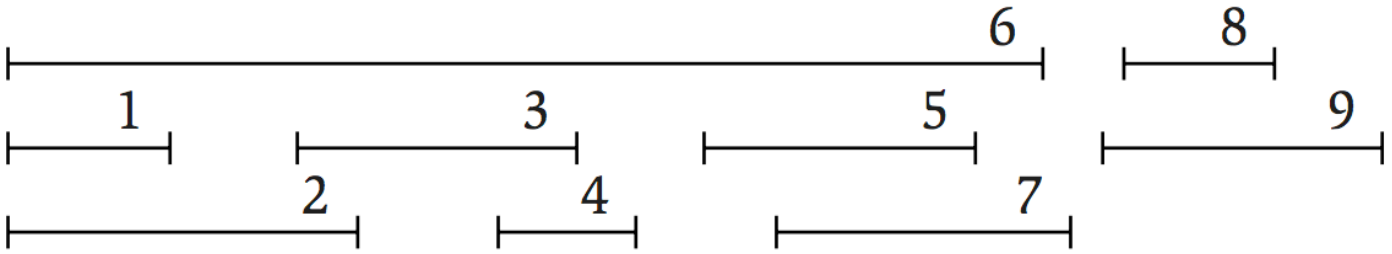
# Interval Scheduling

# (Weighted) Interval Scheduling

- Input: $n$ intervals $(s_i, f_i)$ with values $v_i$
- Output: a compatible schedule $S$ with the largest possible total value
  - A schedule is a subset of intervals $S \subseteq \{1, \dots, n\}$
  - A schedule $S$ is compatible if no two $i, j \in S$ overlap
  - The total value of $S$ is $\sum_{i \in S} v_i$

# (Unweighted) Interval Scheduling

- Input: $n$ intervals $(s_i, f_i)$
- Output: a compatible schedule $S$ with the largest possible size
  - A schedule is a subset of intervals $S \subseteq \{1, \ldots, n\}$
  - A schedule $S$ is compatible if no two $i, j \in S$ overlap



# Possibly Greedy Rules

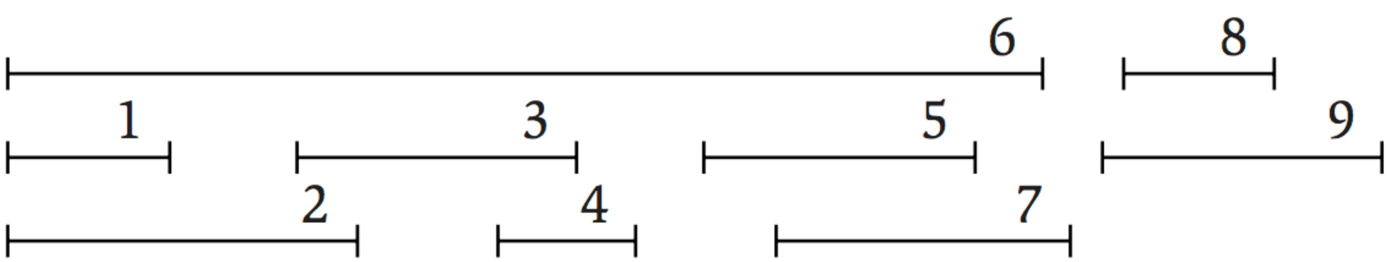
- Choose the shortest interval first

- Choose the interval with earliest start first

- Choose the interval with earliest finish first

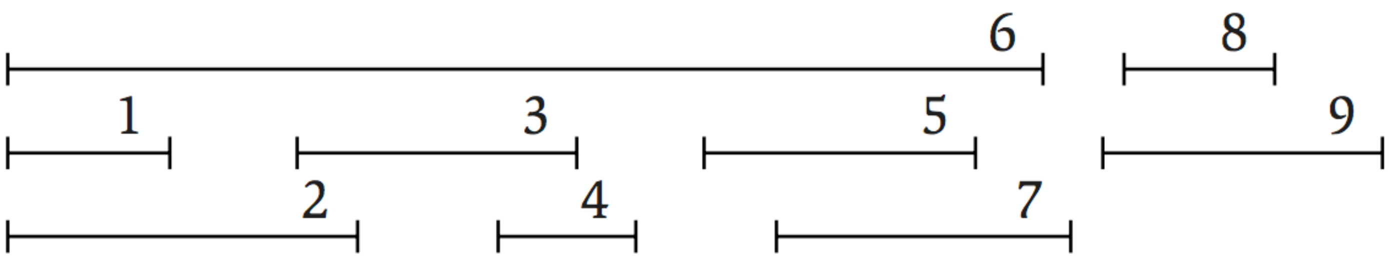
# Greedy Algorithm: Earliest Finish First

- Sort intervals so that $f_1 \leq f_2 \leq \cdots \leq f_n$
- Let $S$ be empty
- For $i = 1, \ldots, n$:
  - If interval $i$ doesn't create a conflict, add $i$ to $S$
- Return $S$



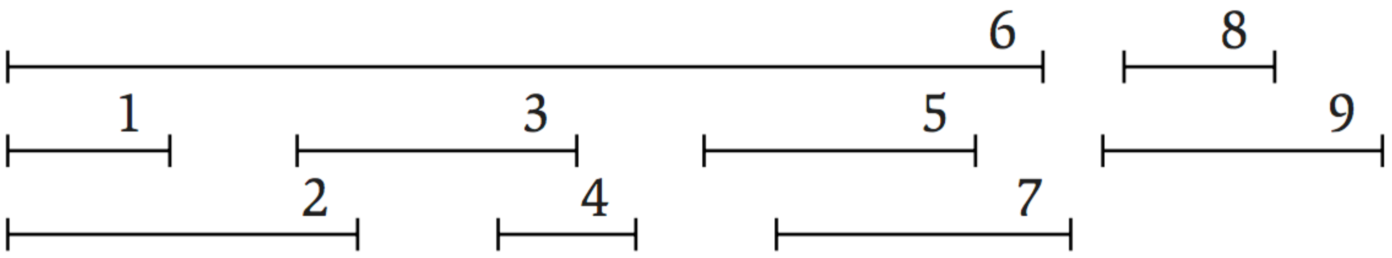
# Greedy Stays Ahead

- How do we know we found an optimal schedule
- “Greedy Stays Ahead” strategy
  - We’ll show that at every point in time, the greedy schedule does better than any other schedule



# Greedy Stays Ahead

- Let $G = \{i_1, \dots, i_r\}$ be greedy's schedule
- Let $O = \{j_1, \dots, j_s\}$ be some optimal schedule
- **Key Claim:** for every $t = 1, \dots, r$, $f_{i_t} \leq f_{j_t}$



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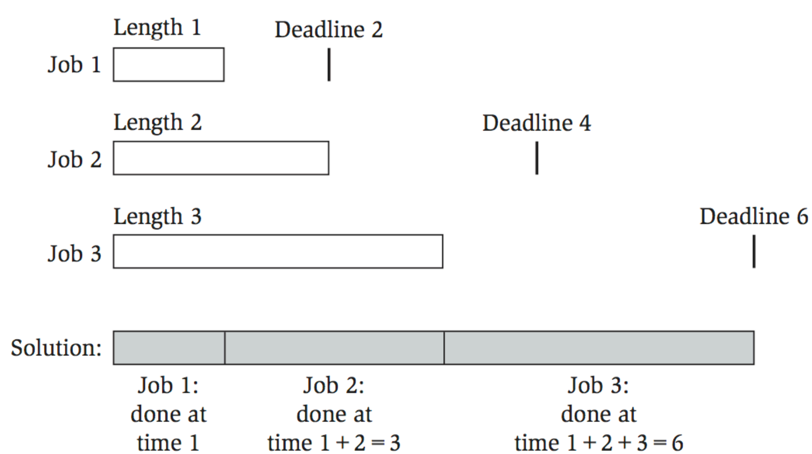
# Minimum Lateness Scheduling

# Minimum Lateness Scheduling

- Input: $n$ jobs with length $t_i$ and deadline $d_i$
  - Simplifying assumption: all deadlines are distinct
- Output: a minimum-lateness schedule for the jobs
  - Can only do one job at a time, no overlap
  - The lateness of job $i$ is $\max\{f_i - d_i, 0\}$
  - The lateness of a schedule is $\max_i\{\max\{f_i - d_i, 0\}\}$



# Possible Greedy Rules

- Choose the shortest job first ($\min t_i$)?

- Choose the most urgent job first ($\min d_i - t_i$)?

# Greedy Algorithm: Earliest Deadline First

- Sort jobs so that $d_1 \leq d_2 \leq \cdots \leq d_n$
- For $i = 1, \ldots, n$:
  - Schedule job $i$ right after job $i - 1$ finishes

# Exchange Argument

- $G$ = greedy schedule, $O$ = optimal schedule

- Exchange Argument:
  - We can transform $O$ to $G$ by exchanging pairs of jobs
  - Each exchange only reduces the lateness of $O$
  - Therefore the lateness of $G$ is at most that of $O$

# Exchange Argument

- $G$ = greedy schedule, $O$ = optimal schedule

- Observation: the optimal schedule has no gaps
  - A schedule is just an ordering of the jobs, with jobs scheduled back-to-back

# Exchange Argument

- $G$ = greedy schedule, $O$ = optimal schedule

- We say that two jobs $i, j$ are inverted in $O$ if $d_i < d_j$ but $j$ comes before $i$
    - Observation: greedy has no inversions

# Exchange Argument

- We say that two jobs $i, j$ are inverted in $O$ if $d_i < d_j$ but $j$ comes before $i$
- Claim: the optimal schedule has no inversions
  - Step 1: suppose $O$ has an inversion, then it has an inversion $i, j$ where $i, j$ are consecutive

# Exchange Argument

- We say that two jobs $i, j$ are inverted in $O$ if $d_i < d_j$ but $j$ comes before $i$
- Claim: the optimal schedule has no inversions
  - Step 1: suppose $O$ has an inversion, then it has an inversion $i, j$ where $i, j$ are consecutive
  - Step 2: if $i, j$ are a consecutive jobs that are inverted then flipping them only reduces the lateness

# Exchange Argument

- If $i, j$ are a consecutive jobs that are inverted then flipping them only reduces the lateness

# Exchange Argument

- We say that two jobs $i, j$ are inverted in $O$ if $d_i < d_j$ but $j$ comes before $i$
- Claim: the optimal schedule has no inversions
  - Step 1: suppose $O$ has an inversion, then it has an inversion $i, j$ where $i, j$ are consecutive
  - Step 2: if $i, j$ are a consecutive jobs that are inverted then flipping them only reduces the lateness

- $G$ is the unique schedule with no inversions, $O$ is the unique schedule with no inversions, $G = O$