CS3000: Algorithms & Data
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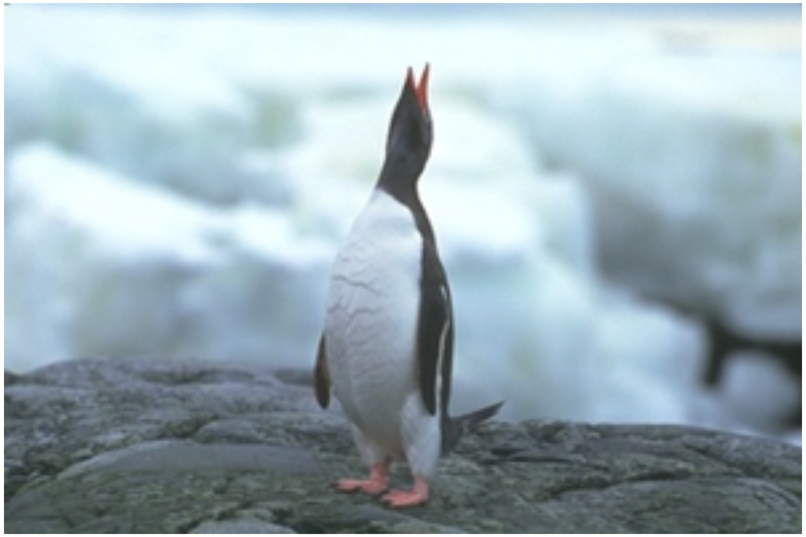
Lecture 17:
More Applications of Network Flow

March 25, 2020

Image Segmentation

Image Segmentation



- Separate image into foreground and background
- We have some idea of:
  - whether pixel i is in the foreground or background
  - whether pair (i,j) are likely to go together

Image Segmentation

- Input:
  - an undirected graph $G = (V, E)$; V = “pixels”, E = “pairs”
  - likelihoods $a_i, b_i \geq 0$ for every $i \in V$
  - separation penalty $p_{ij} \geq 0$ for every $(i, j) \in E$
- Output:
  - a partition of V into (A, B) that maximizes

$$q(A, B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ \text{btw } A \text{ and } B}} p_{ij}$$

Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG asks us to maximize, MINCUT asks us to minimize

$$\max_{A,B} \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ \text{btw } A \text{ and } B}} p_{ij} \quad \Leftrightarrow \quad \min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ \text{btw } A \text{ and } B}} p_{ij}$$

Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG allows any partition, MINCUT requires $s \in A$, $t \in B$

Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG has edges **between A and B**, MINCUT considers edges **from A to B**

$$\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ \text{btw } A \text{ and } B}} p_{ij}$$

$$\min_{A,B} \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij}$$

Reduction to MinCut

- Differences between SEG and MINCUT:
  - SEG has terms for each node in A,B, MINCUT only has terms for edges from A to B

$$\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ \text{btw } A \text{ and } B}} p_{ij}$$

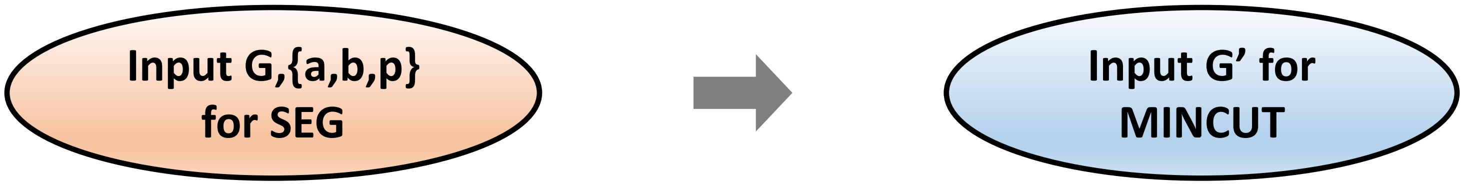
$$\min_{A,B} \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij}$$

Reduction to MinCut

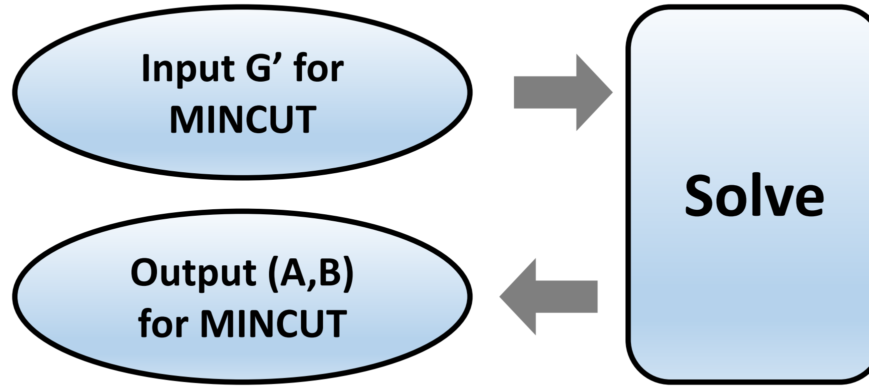
- How should the reduction work?
  - capacity of the cut should correspond to the function we're trying to minimize

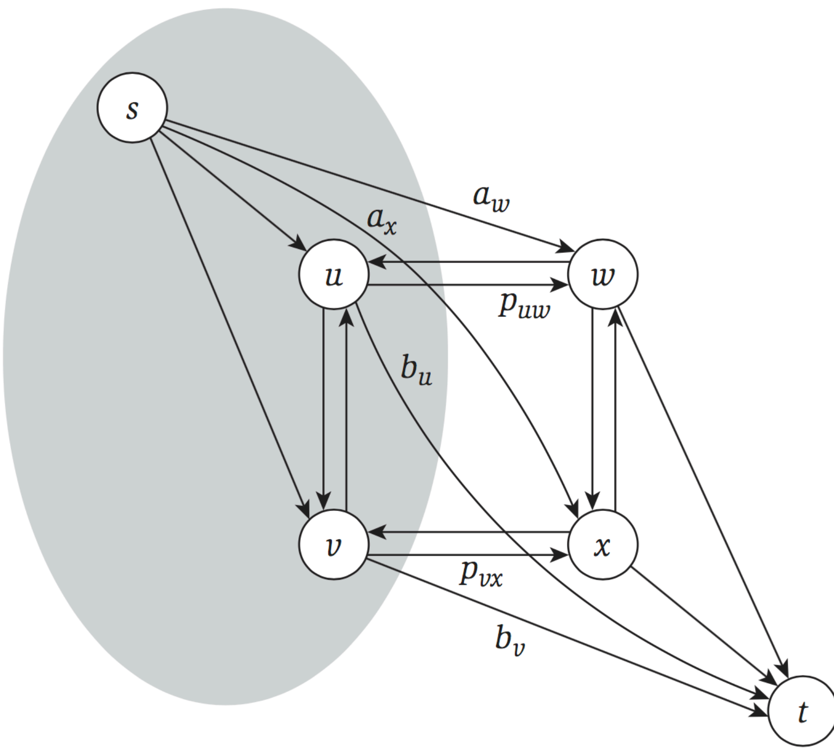
$$\min_{A,B} \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ \text{from } A \text{ to } B}} p_{ij}$$

Step 1: Transform the Input



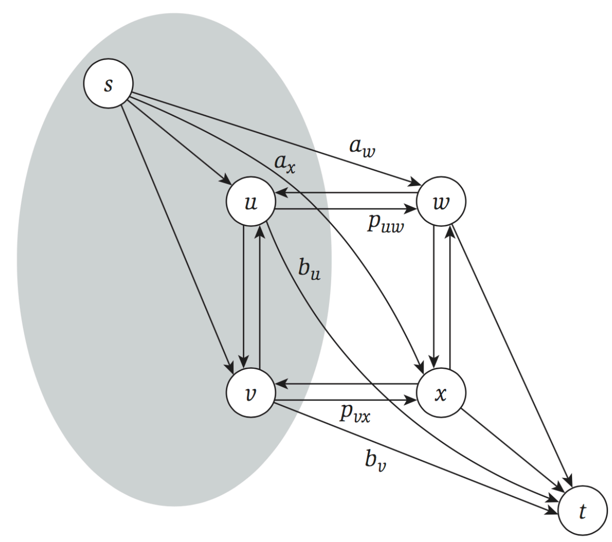
Step 2: Receive the Output





Step 3: Transform the Output

Output (A,B) for SEG ⟵ **Output (A,B) for MINCUT**



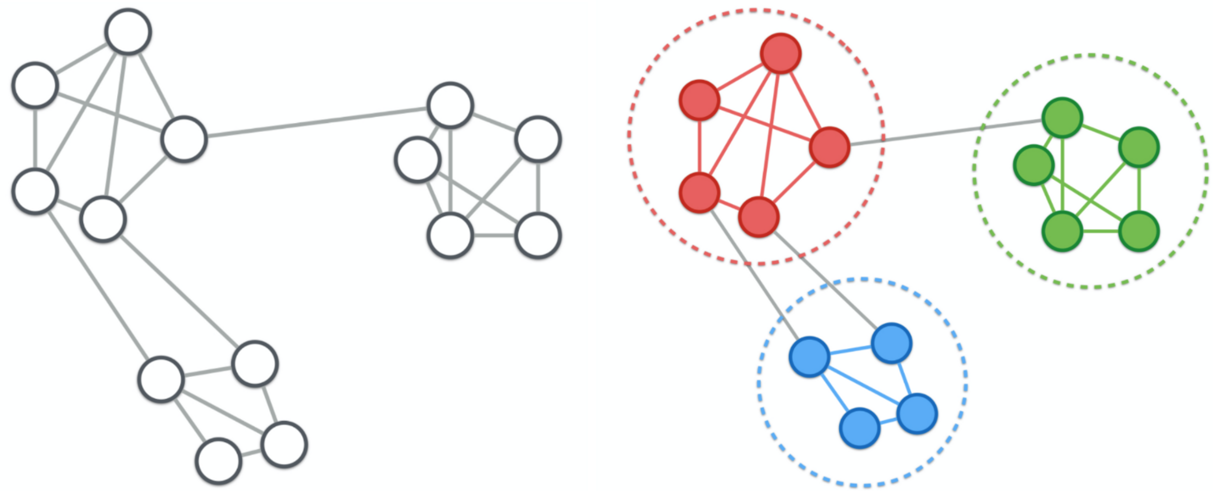
Reduction to MinCut

- correctness?

- running time?

Densest Subgraph

Image Segmentation



- Want to identify communities in a network
  - "Community": a set of nodes that have a lot of connections inside and few outside

Densest Subgraph

- Input:
  - an undirected graph $G = (V, E)$
- Output:
  - a subset of nodes $A \subseteq V$ that maximizes $\frac{2|E(A,A)|}{|A|}$

Reduction to MinCut

- Different objectives
  - maximize $\frac{2|E(A,A)|}{|A|}$ vs. minimize $|E(A,B)|$

- Suppose $\frac{2|E(A,A)|}{|A|} \geq \delta$ and see what that implies

$$\Leftrightarrow 2|E(A,A)| \geq \delta|A|$$

$$\Leftrightarrow \Sigma_{v\in A} \deg(v) \; - |E(A,B)| \geq \delta|A|$$

$$\Leftrightarrow \Sigma_{v\in V} \deg(v) \; - \Sigma_{v\in B} \deg(v) - |E(A,B)| \geq \delta|A|$$

$$\Leftrightarrow 2|E| \; - \Sigma_{v\in B} \deg(v) - |E(A,B)| \geq \delta|A|$$

$$\Leftrightarrow \Sigma_{v\in B} \deg(v) + \delta|A| + \; |E(A,B)| \leq 2|E|$$

$$\Leftrightarrow \Sigma_{v\in B} \deg(v) + \Sigma_{v\in A}\delta + \Sigma_{e \text{ from } A \text{ to } B} \; 1 \leq 2|E|$$

Reduction to MinCut

$$\Sigma_{v \in B} \deg(v) + \Sigma_{v \in A} \delta + \Sigma_{e \text{ from } A \text{ to } B} 1 \leq 2|E|$$