CS3000: Algorithms & Data
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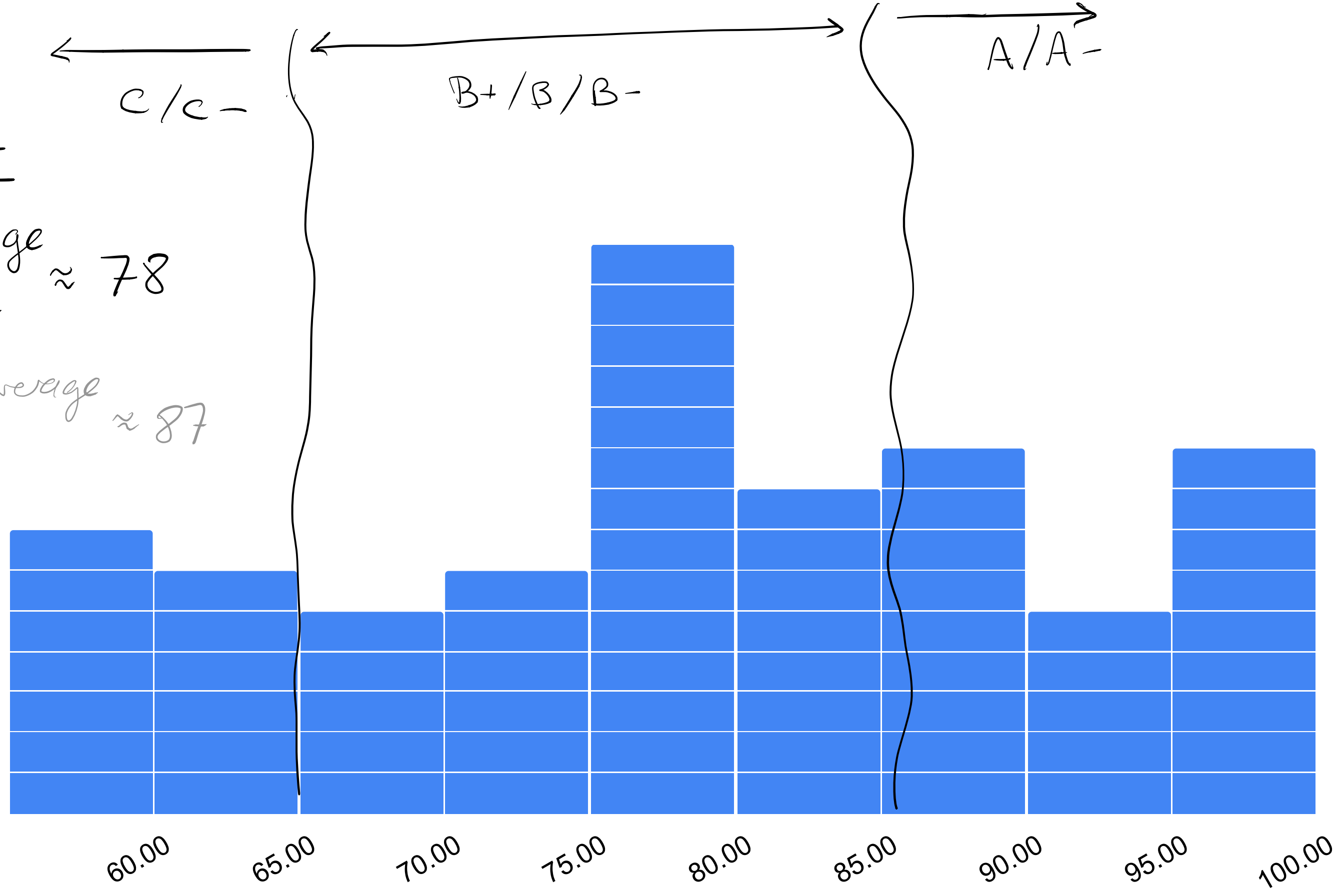
Lecture 10:

- Graphs
- Graph Traversals: DFS
- Topological Sort

Feb 19, 2020

Midterm 1

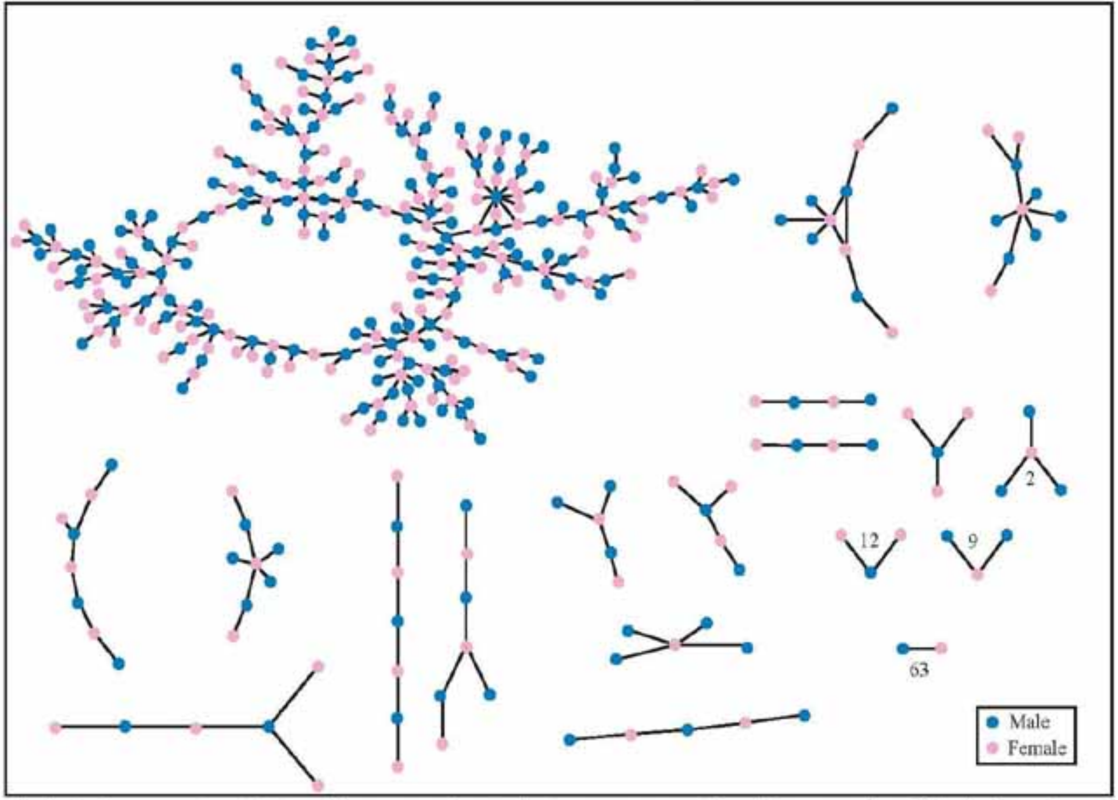


MTI
Average
Score ≈ 78

HW Average
Score ≈ 87

What’s Next

The Structure of Romantic and Sexual Relations at "Jefferson High School"



Each circle represents a student and lines connecting students represent romantic relations occuring within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (i.e. we found 63 pairs unconnected to anyone else).

What’s Next

- **Graph Algorithms:**
  - **Graphs:** Key Definitions, Properties, Representations
  - **Exploring Graphs:** Breadth/Depth First Search
    - Applications: Connectivity, Bipartiteness, Topological Sorting
  - **Shortest Paths:**
    - Dijkstra
    - Bellman-Ford (Dynamic Programming)
  - **Minimum Spanning Trees:**
    - Borůvka, Prim, Kruskal
  - **Network Flow:**
    - Algorithms
    - Reductions to Network Flow

Graphs

Graphs: Key Definitions

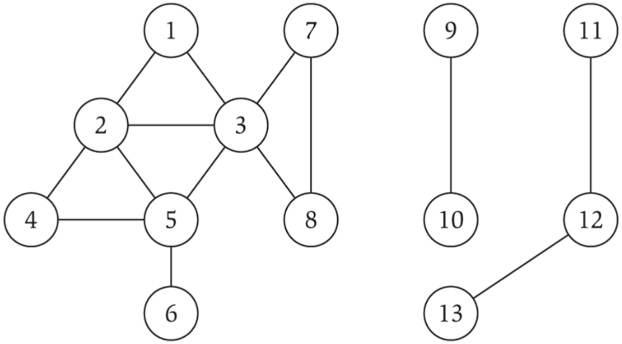
$n = |V|$ # of nodes

$m = |E|$ # of edges

$\deg(u) =$ # of neighbors

- **Definition:** A directed graph $G = (V, E)$
  - V is the set of nodes/vertices
  - $E \subseteq V \times V$ is the set of edges
  - An edge is an ordered $e = (u, v)$ "from u to v"
- **Definition:** An undirected graph $G = (V, E)$
  - Edges are unordered $e = (u, v)$ "between u and v"

- **Simple Graph:**
  - No duplicate edges
  - No self-loops $e = (u, u)$

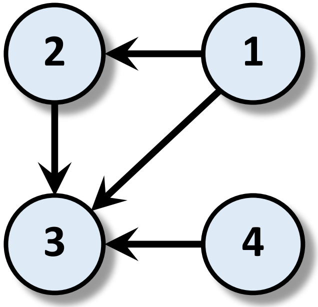


Adjacency Matrices

- The adjacency matrix of a graph $G = (V, E)$ with n nodes is the matrix $A[1{:}n, 1{:}n]$ where

$$A[i,j] = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0



Cost

Space: $\Theta(V^2)$

Lookup: $\Theta(1)$ time

List Neighbors: $\Theta(V)$ time

Adjacency Lists (Undirected)

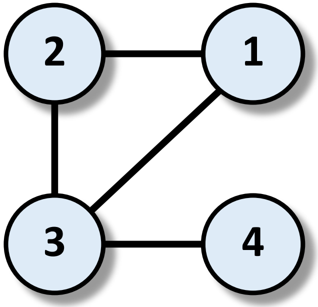
- The adjacency list of a vertex $v \in V$ is the list $A[v]$ of all u s.t. $(v, u) \in E$

$A[1] = \{2,3\}$

$A[2] = \{1,3\}$

$A[3] = \{1,2,4\}$

$A[4] = \{3\}$



Adjacency Lists (Directed)

- The adjacency list of a vertex $v \in V$ are the lists
  - $A_{out}[v]$ of all u s.t. $(v, u) \in E$
  - $A_{in}[v]$ of all u s.t. $(u, v) \in E$

Space: $\Theta(n+m)$

List Neighbors of Node u: $O(\deg(u)+1)$

Lookup Edge (u,v): $O(\deg(u)+1)$

$A_{out}[1] = \{2,3\}$

$A_{out}[2] = \{3\}$

$A_{out}[3] = \{\ \}$

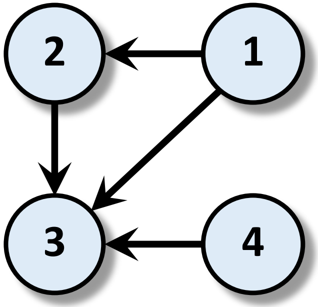
$A_{out}[4] = \{3\}$

$A_{in}[1] = \{\ \}$

$A_{in}[2] = \{1\}$

$A_{in}[3] = \{1,2,4\}$

$A_{in}[4] = \{\ \}$



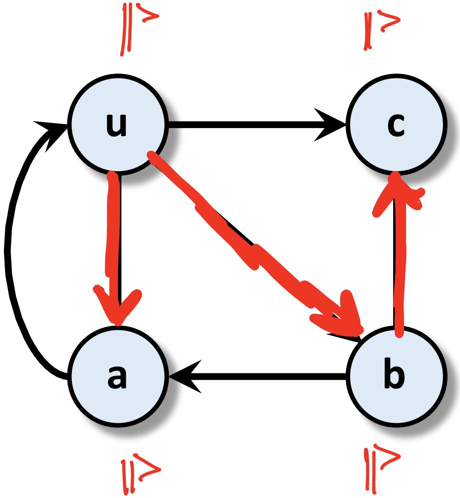
Depth-First Search (DFS)

Depth-First Search

```
G = (V,E) is a graph
explored[u] = 0 ∀u

DFS(u):
  explored[u] = 1

  for ((u,v) in E):
    if (explored[v]=0):
      parent[v] = u
      DFS(v)
```



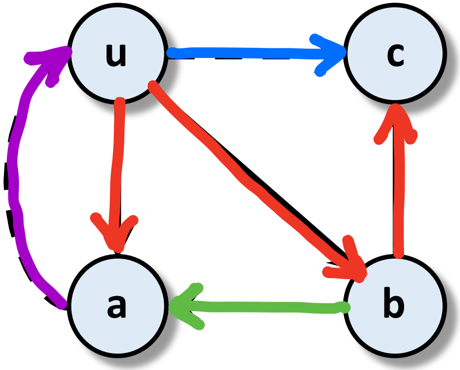
Red arrows give a path from u to each other node

Running Time: $O(n+m)$

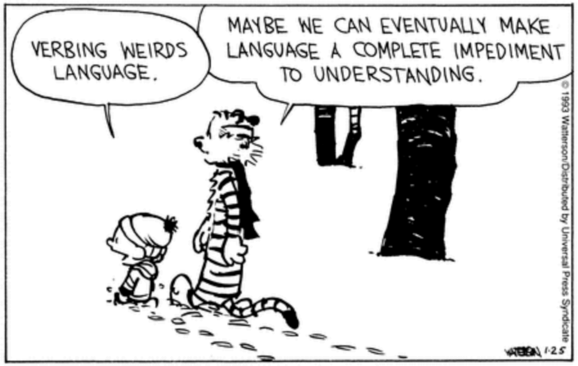
(In the graph reachable from u)

Depth-First Search

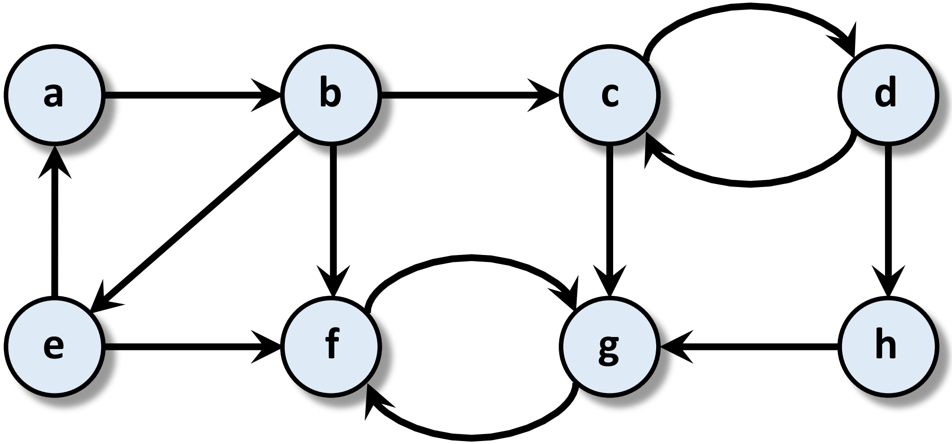
- **Fact:** The parent-child edges form a (directed) tree
- **Each edge has a type:**
  - **Tree edges:** $(u, a), (u, b), (b, c)$ 🟥
    - These are the edges that explore new nodes
  - **Forward edges:** (u, c) 🟦
    - Ancestor to descendant
  - **Backward edges:** (a, u) 🟪
    - Descendant to ancestor
    - Implies a directed cycle!
  - **Cross edges:** (b, a) 🟩
    - No ancestral relation



Ask the Audience



- DFS starting from node a
  - Search in alphabetical order
  - Label edges with {tree,forward,backward,cross}



Connected Components

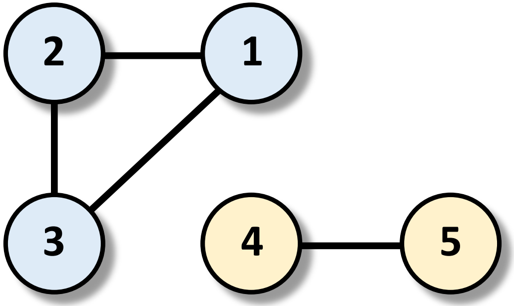
Paths/Connectivity

- A path is a sequence of consecutive edges in E
  - $P = u - w_1 - w_2 - w_3 - \cdots - w_{k-1} - v$
  - The length of the path is the # of edges

- An undirected graph is connected if for every two vertices $u, v \in V$, there is a path from u to v
- A directed graph is strongly connected if for every two vertices $u, v \in V$, there are paths from u to v and from v to u

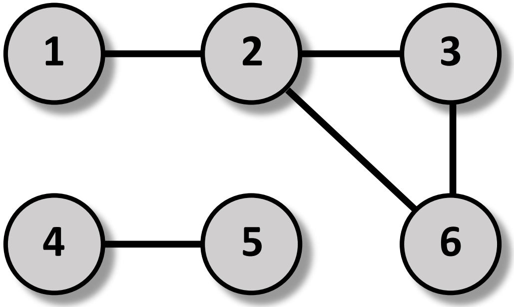
Connected Components (Undirected)

- **Problem:** Given an undirected graph G, split it into connected components
- **Input:** Undirected graph $G = (V, E)$
- **Output:** A labeling of the vertices by their connected component



Connected Components (Undirected)

- **Algorithm:**
  - Pick a node v
  - Use DFS to find all nodes reachable from v
  - Labels those as one connected component
  - Repeat until all nodes are in some component



Connected Components (Undirected)

```
CC(G = (V,E)):
  // Initialize an empty array and a counter
  let comp[1:n] ←⊥, c ← 1

  // Iterate through nodes
  for (u = 1,…,n):
    // Ignore this node if it already has a comp.
    // Otherwise, explore it using DFS
    if (comp[u] != ⊥):
      run DFS(G,u)
      let comp[v] ← c for every v found by DFS
      let c ← c + 1

  output comp[1:n]
```

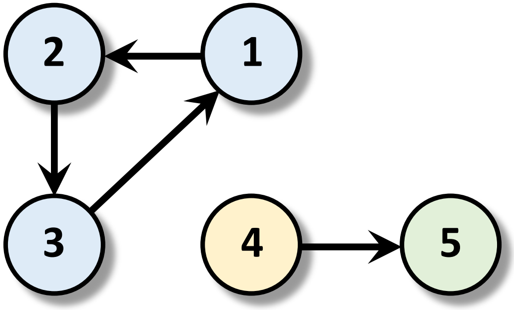
Running Time

Connected Components (Undirected)

- **Problem:** Given an undirected graph G, split it into connected components
- **Algorithm:** Can split a graph into conneted components in time $O(n+m)$ using DFS
- **Punchline:** Usually assume graphs are connected
    - Implicitly assume that we have already broken the graph into CCs in $O(n+m)$ time

Strong Components (Directed)

- **Problem:** Given a directed graph G, split it into strongly connected components
- **Input:** Directed graph $G = (V, E)$
- **Output:** A labeling of the vertices by their strongly connected component



Strong Components (Directed)

- **Observation:** $\mathrm{SCC}(s)$ is all nodes $v \in V$ such that v is reachable from s and vice versa
  - Can find all nodes reachable from s using BFS
  - How do we find all nodes that can reach s?

Strong Components (Directed)

```
SCC(G = (V,E)):
  let $G^R$ be G with all edges “reversed”

  // Initialize an array and counter
  let comp[1:n] ← ⊥, c ← 1

  for (u = 1,…,n):
    // If u has not been explored
    if (comp[u] != ⊥):
      let S be the nodes found by DFS(G,u)
      let T be the nodes found by DFS($G^R$,u)
      // S ∩ T contains SCC(u)
      label S ∩ T with c
      let c ← c + 1

  return comp
```

Strong Components (Directed)

- **Problem:** Given a directed graph G, split it into strongly connected components
- **Input:** Directed graph $G = (V, E)$
- **Output:** A labeling of the vertices by their strongly connected component

- Find SCCs in $O(n^2 + nm)$ time using DFS
- **Can find SCCs in $\boldsymbol{O(n + m)}$ time** using a more clever version of DFS

Post-Ordering

Post-Ordering

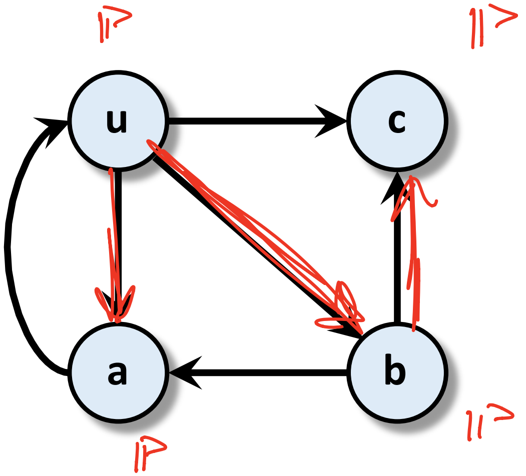
clock
4

```
G = (V,E) is a graph
explored[u] = 0 ∀u

DFS(u):
  explored[u] = 1

  for ((u,v) in E):
    if (explored[v]=0):
      parent[v] = u
      DFS(v)

  post-visit(u)
```



Vertex	Post-Order
a	1
c	2
b	3
u	4

- Maintain a counter **`clock`**, initially set **`clock = 1`**
- **`post-visit(u):`**
  `set postorder[u]=clock, clock=clock+1`

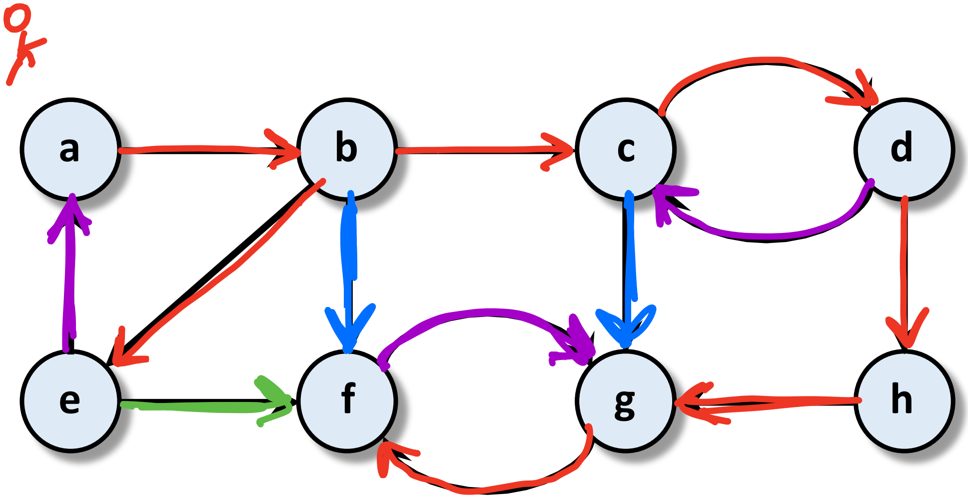
Example

clock

~~||||~~ ||||

🟥 tree
🟦 forward
🟪 backward
🟩 cross

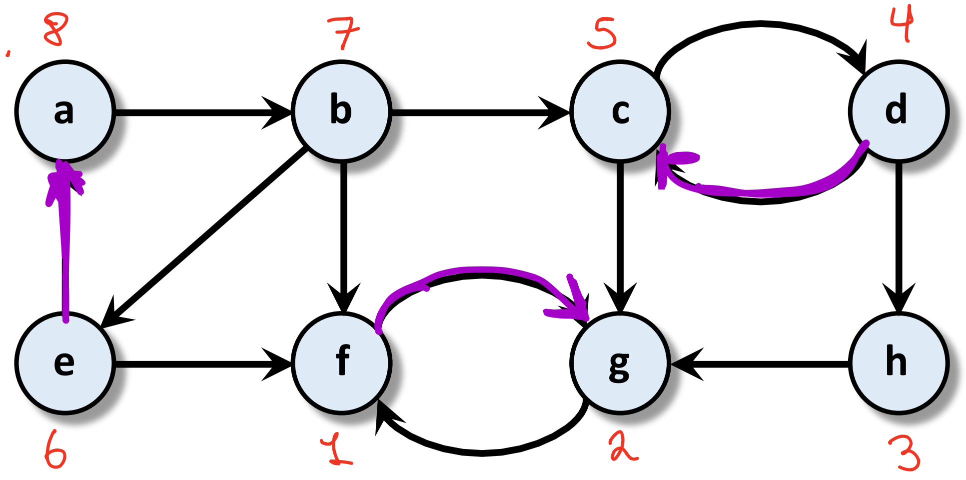
- Compute the **post-order** of this graph
  - DFS from a, search in alphabetical order



Vertex	a	b	c	d	e	f	g	h
Post-Order	8	7	5	4	6	1	2	3

Example

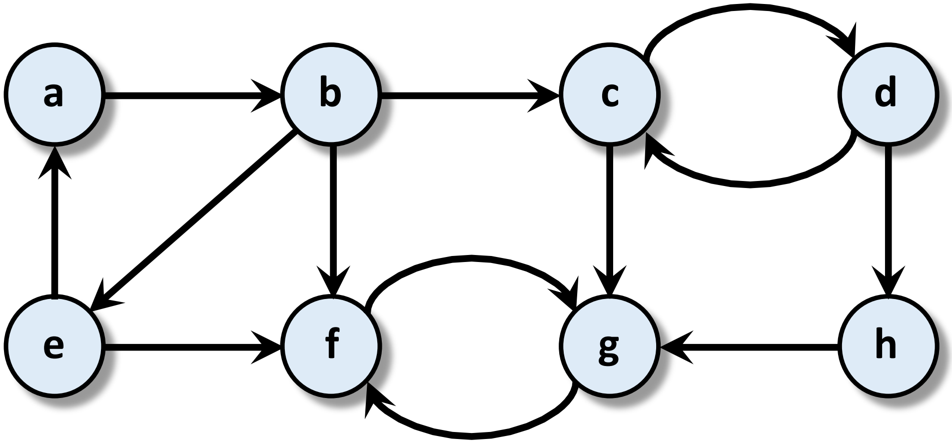
- Compute the **post-order** of this graph
  - DFS from a, search in alphabetical order



Vertex	a	b	c	d	e	f	g	h
Post-Order	8	7	5	4	6	1	2	3

Obervation

- **Observation:** if postorder[u] < postorder[v] then (u,v) is a backward edge

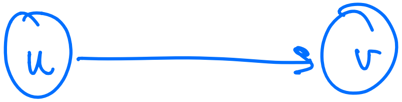


Vertex	a	b	c	d	e	f	g	h
Post-Order	8	7	5	4	6	1	2	3

Observation

Gives an algorithm for finding backwards edges / cycles

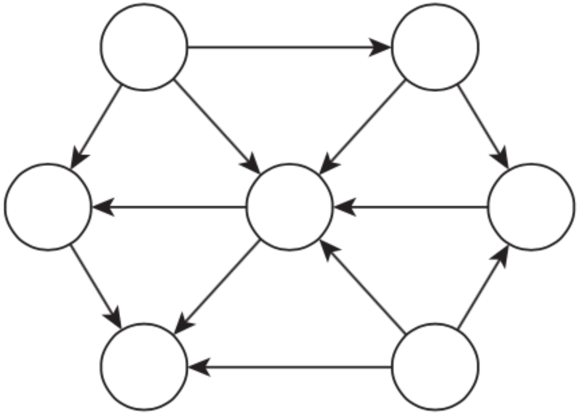
- **Observation:** if postorder[u] < postorder[v] then (u,v) is a backward edge
  - DFS(u) can't finish until its children are finished
  - If postorder[u] < postorder[v], then DFS(u) finishes before DFS(v), thus DFS(v) is not called by DFS(u)
  - When we ran DFS(u), we must have had explored[v]=1
    - Thus, DFS(v) started before DFS(u)
  - DFS(v) started before DFS(u) but finished after
    - Can only happen for a backward edge



Topological Ordering

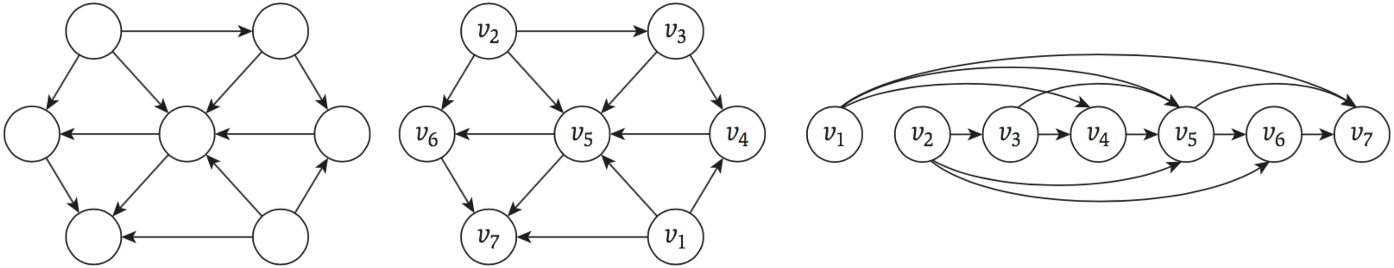
Directed Acyclic Graphs (DAGs)

- **DAG:** A directed graph with no directed cycles
- Can be much more complex than a forest



Directed Acyclic Graphs (DAGs)

- **DAG:** A directed graph with no directed cycles
- DAGs represent precedence relationships



- A topological ordering of a directed graph is a labeling of the nodes from $v_1, \ldots, v_n$ so that all edges go "forwards", that is $(v_i, v_j) \in E \Rightarrow j > i$
  - G has a topological ordering $\Rightarrow$ G is a DAG

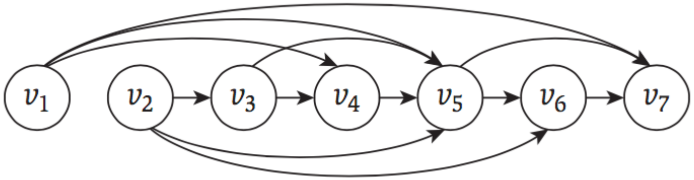
Directed Acyclic Graphs (DAGs)

- **Problem 1:** given a digraph G, is it a DAG?
- **Problem 2:** given a digraph G, can it be topologically ordered?

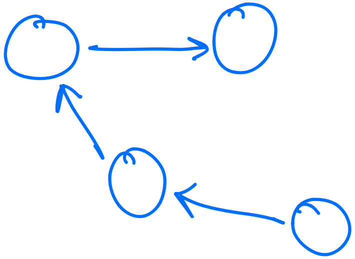
- **Thm:** G has a topological ordering $\Leftrightarrow$ G is a DAG
  - We will design one algorithm that either outputs a topological ordering or finds a directed cycle

Topological Ordering

- **Observation:** the first node must have no in-edges

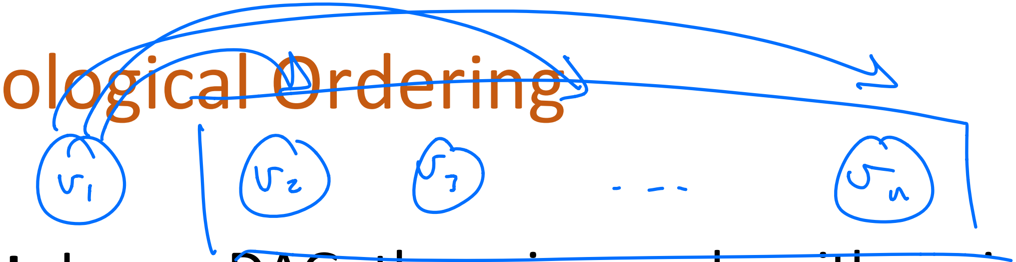


- **Observation:** In any DAG, there is always a node with no incoming edges



- Follow incoming edges until either you find a node w/o any, or find a cycle

Topological Ordering



- **Fact:** In any DAG, there is a node with no incoming edges

For every $n \in \mathbb{N}$, every DAG with n nodes has a top. order

- **Thm:** Every DAG has a topological ordering
- **Proof (Induction):**

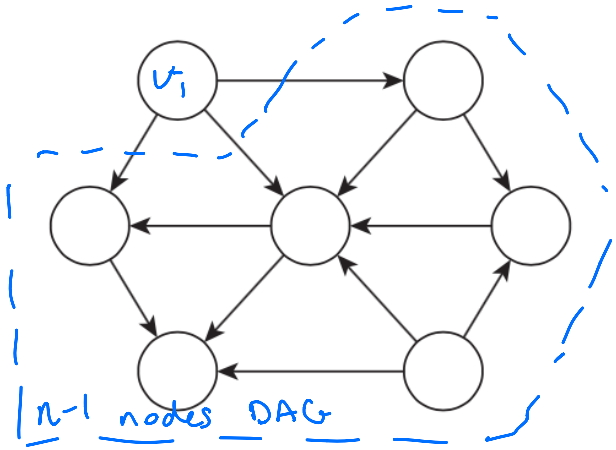
Base Case: $n = 1$ is obvious

Inductive Step: Suppose every $n-1$ node DAG has a top. ordering

① Choose a node w/ no incoming edges

② Remove v_1 and its edges

③ By induction the remainder has a top. ordering $v_2, v_3, \dots, v_n$



Faster Topological Ordering

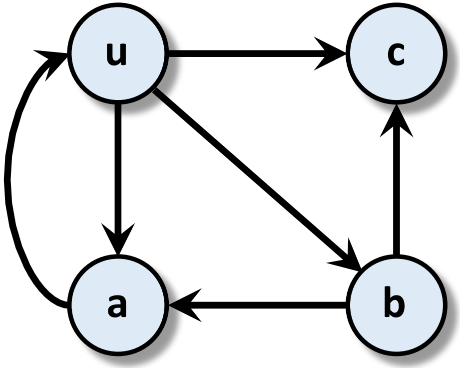
Post-Ordering

```
G = (V,E) is a graph
explored[u] = 0 ∀u

DFS(u):
  explored[u] = 1

  for ((u,v) in E):
    if (explored[v]=0):
      parent[v] = u
      DFS(v)

  post-visit(u)
```



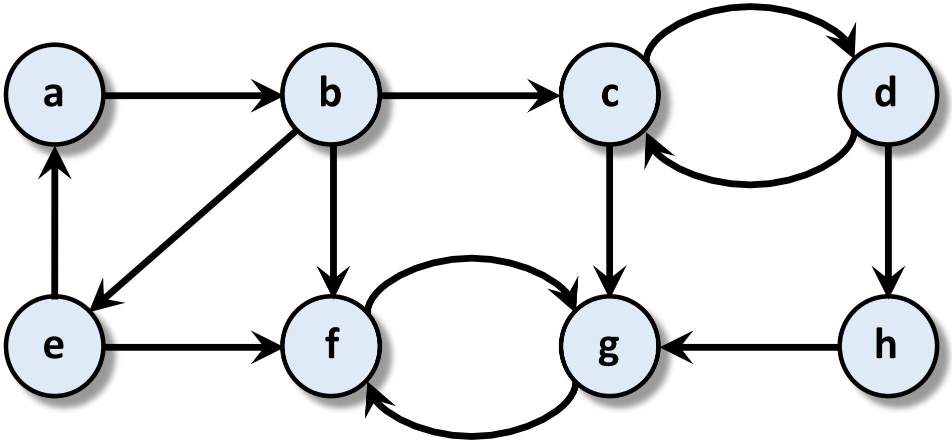
Vertex	Post-Order

- Maintain a counter `clock`, initially set `clock = 1`
- `post-visit(u):`

  `set postorder[u]=clock, clock=clock+1`

Example

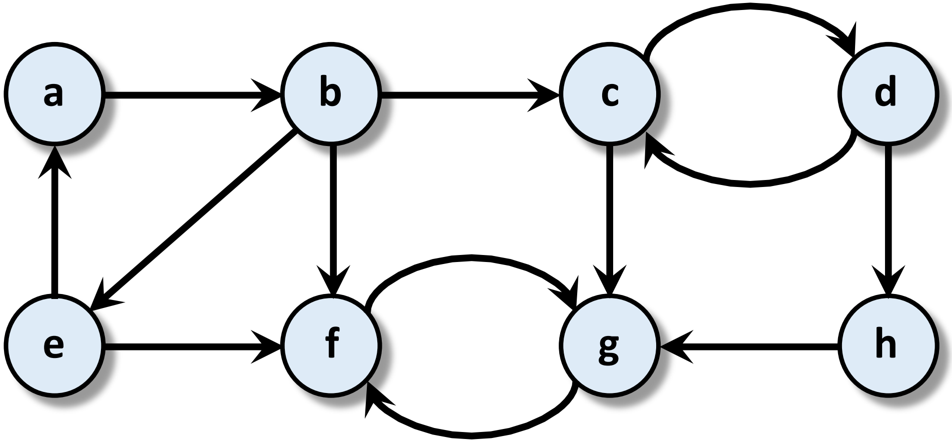
- Compute the **post-order** of this graph
  - DFS from a, search in alphabetical order



Vertex	a	b	c	d	e	f	g	h
Post-Order								

Example

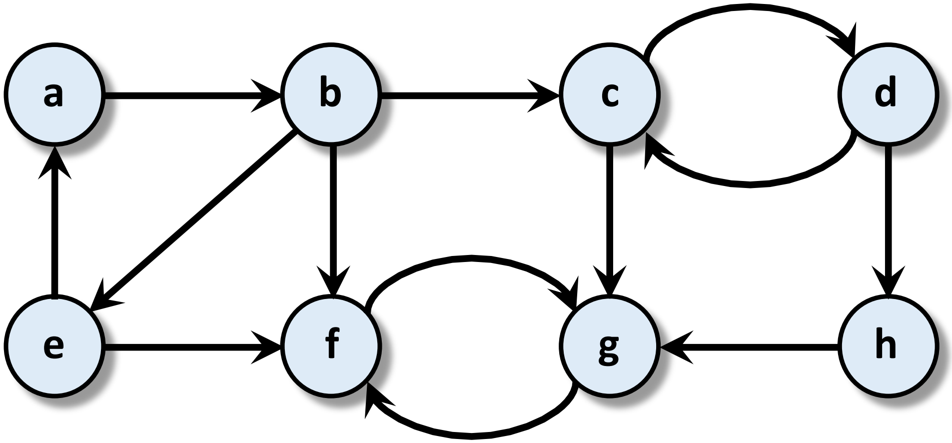
- Compute the **post-order** of this graph
  - DFS from a, search in alphabetical order



Vertex	a	b	c	d	e	f	g	h
Post-Order	8	7	5	4	6	1	2	3

Obervation

- **Observation:** if postorder[u] < postorder[v] then (u,v) is a backward edge



Vertex	a	b	c	d	e	f	g	h
Post-Order	8	7	5	4	6	1	2	3

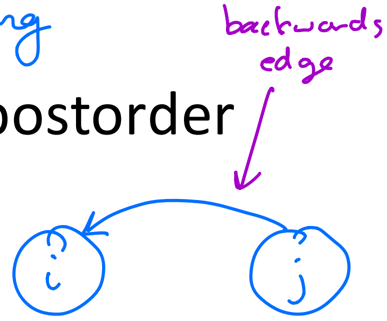
Observation

- **Observation:** if postorder[u] < postorder[v] then (u,v) is a backward edge
  - DFS(u) can't finish until its children are finished
  - If postorder[u] < postorder[v], then DFS(u) finishes before DFS(v), thus DFS(v) is not called by DFS(u)
  - When we ran DFS(u), we must have had explored[v]=1
    - Thus, DFS(v) started before DFS(u)
  - DFS(v) started before DFS(u) but finished after
    - Can only happen for a backward edge

Fast Topological Ordering

The post-order is a backwards top. ordering

- **Claim:** ordering nodes by decreasing postorder gives a topological ordering



postorder[j] < postorder[i]

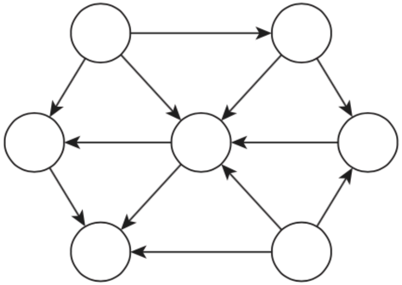
$j > i$

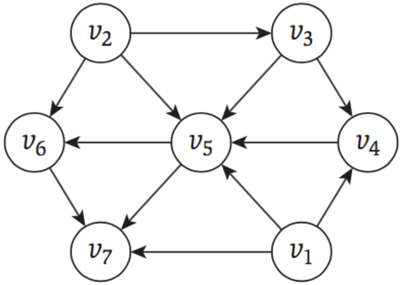
$(j, i) \in E$

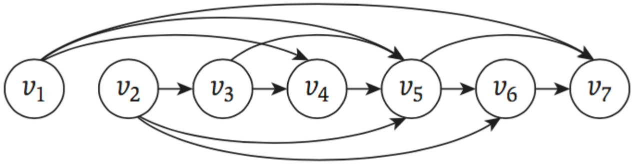
- **Proof:**
  - A DAG has no backward edges
  - Suppose this is **not** a topological ordering
    - That means there exists an edge (u,v) such that postorder[u] < postorder[v]
    - We showed that any such (u,v) is a backward edge
    - But there are no backward edges, contradiction!

Topological Ordering (TO)

- **DAG:** A directed graph with no directed cycles
- Any DAG can be **toplogically ordered**
  - Label nodes $v_1, \dots, v_n$ so that $(v_i, v_j) \in E \Longrightarrow j > i$







- Can compute a TO in $O(n + m)$ time using DFS
  - Reverse of post-order is a topological order

Designing the Algorithm

- **Claim:** If BFS fails, then G contains an odd cycle
  - If G contains an odd cycle then G can't be 2-colored!
  - Example of a phenomenon called **duality**