CS3000: Algorithms & Data
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Lecture 22:
- Online Learning

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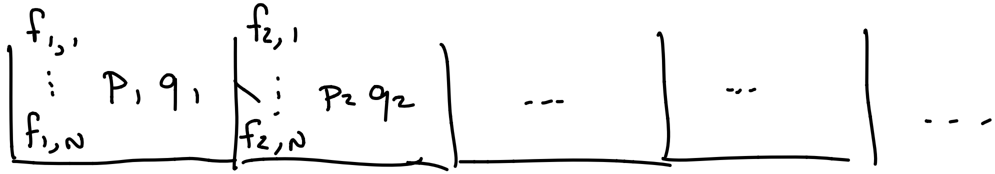
Picking Good Experts

- Suppose you have N "experts" making predictions
  - Weather forecasters
  - Financial advisors
  - Recommender systems
  - …
- Most of them are bad, but one might be good!
- Who's predictions should we trust?

Model



- There are T time periods, in each:
  - experts make predictions $f_{t,1}, \dots, f_{t,N} \in \{0,1\}$ binary outcomes ("rain"/"sun")
  - you have to make a decision $p_t \in \{0,1\}$
  - some outcome $q_t \in \{0,1\}$ is revealed
  - a "mistake" is when $p_t \neq q_t$
- **Goal:** minimize mistakes
  - The # of mistakes we make is M
  - The best expert in hindsight makes M^* mistakes
  - Want to ensure that $M \approx M^*$

Level I: The Halving Algorithm (HA)

- Assumption: some expert makes 0 mistakes

Let $C_1 \leftarrow \{1, \ldots, n\}$
For $t = 1, \ldots, T$:
    Let p_t be the majority vote of experts in C_t
    Let C_{t+1} be the experts with no mistakes so far

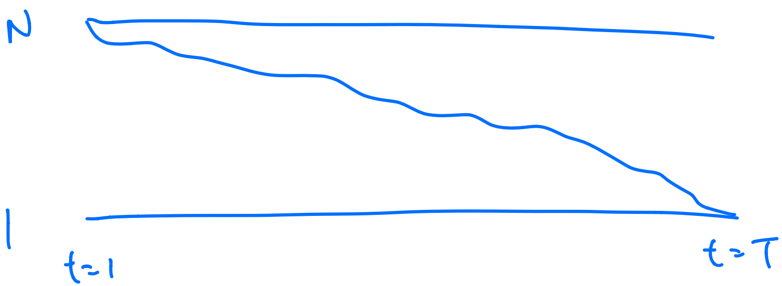
Level I: The Halving Algorithm

$M^* = 0$

- **Thm:** If some expert makes 0 mistakes then HA makes $\leq \log_2 N$ mistakes.

Measure of progress: $|C_t|$

Fact: $N = |C_1| \geq |C_2| \geq \ldots \geq |C_T| \geq 1$



If we make a mistake on day t, then $|C_{t+1}| \leq \frac{|C_t|}{2}$

$\therefore \quad |C_T| \leq N \cdot 2^{-M}$

Key Equation:

$$1 \le |C_T| \le N \cdot 2^{-M}$$

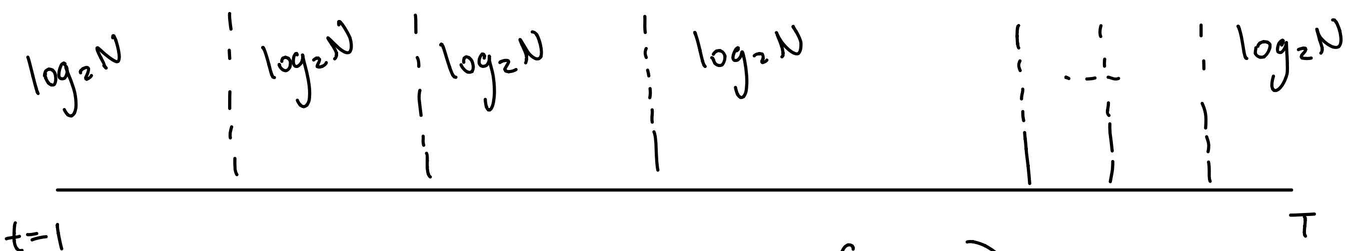
$$\Rightarrow \quad 1 \le N \cdot 2^{-M}$$

$$2^M \le N$$

$$M \le \log_2 N \qquad \blacksquare$$

Level II: Repeated Halving (RHA)

Let $C_1 \leftarrow \{1, \ldots, n\}$
For $t = 1, \ldots, T$:
  Let p_t be the majority vote of experts in C_t
  Let C_{t+1} be the experts with no mistakes so far
  If $C_{t+1} = \emptyset$, let $C_{t+1} = \{1, \ldots, n\}$



Best expert makes M^* mistakes

(M^*+1) intervals
$\times$ $\log_2 N$ mistakes / interval

$M \le (M^*+1)\log_2 N$

Level II: Repeated Halving

- **Thm:** If some expert makes $\leq M$ mistakes then RHA makes $\leq (M+1)(\log_2 N + 1)$ mistakes.

- Worse than the best expert by a factor of $\log_2 N$

Level III: Weighted Majority (WM)

measure of progress

Give each expert a weight $w_{t,i} \leftarrow 1$ $\quad W_t = \sum_{i=1}^{N} w_{t,i}$ $\quad W_1 = N$

For $t = 1, \ldots, T$:

Let p_t be the weighted majority vote of experts

For $i = 1, \ldots, N$:

If (expert i made a mistake): $w_{t+1,i} \leftarrow \frac{w_{t,i}}{2}$

Else: $w_{t+1,i} \leftarrow w_{t,i}$

$\sum_{i: f_{t,i}=1} w_{t,i}$ vs. $\sum_{i: f_{t,i}=0} w_{t,i}$

Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M^* + \log_2 N)$ mistakes.

- Proof:
  - measure of progress is $W_t = \sum_{i=1}^{N} w_{t,i}$
  - if we make M mistakes, then $W_T \leq N \cdot \left(\frac{3}{4}\right)^M$
  - if the best expert i^* makes M^* mistakes, then

$$W_T = \sum_{i=1}^{N} w_{T,i} \geq w_{T,i^*} = \left(\frac{1}{2}\right)^{M^*}$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^{M^*} \leq W_T \leq N \cdot \left(\frac{3}{4}\right)^M$$

$$\left(\tfrac{1}{2}\right)^{M^*} \le N \cdot \left(\tfrac{3}{4}\right)^{M} \qquad \text{(take } \log_2 \text{ of both sides)}$$

$$M^* \log_2\left(\tfrac{1}{2}\right) \le M \log_2\left(\tfrac{3}{4}\right) + \log_2 N$$

$$-M^* \log_2 2 \le -M \log_2\left(\tfrac{4}{3}\right) + \log_2 N$$

$$M \le \frac{M^* \cancel{\log_2 2} + \log_2 N}{\log_2\left(\tfrac{4}{3}\right)}$$

$$M \le 2.4\left(M^* + \log_2 N\right) \qquad \blacksquare$$

Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M^* + \log_2 N)$ mistakes.

Some expert has wt $\geq 2^{-M^*}$

Every mistake decreases total wt by a factor of $\left(\frac{1}{4}\right)$

$$2^{-M^*} \leq W_T \leq N \cdot \left(\frac{3}{4}\right)^M$$

$$-M^* \leq \log_2 N + M \log_2\left(\frac{3}{4}\right)$$

$$M \leq \left(M^* + \log_2 N\right) \frac{-1}{\log_2(3/4)}$$

$$= \frac{1}{\log_2(4/3)} \approx 2.4$$

Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M + \log_2 N)$ mistakes.

Some expert has wt $\geq (1-\varepsilon)^{M^*}$

Every mistake decreases total wt by a factor of $\left(1-\frac{\varepsilon}{2}\right)$

$$(1-\varepsilon)^{M^*} \leq N \cdot \left(1-\frac{\varepsilon}{2}\right)^{M}$$

$$M^* \cdot \ln(1-\varepsilon) \leq \ln N + M \cdot \ln\left(1-\frac{\varepsilon}{2}\right)$$

$$-M^* \cdot \ln\left(\frac{1}{1-\varepsilon}\right) \leq \ln N - M \cdot \ln\left(\frac{1}{1-\varepsilon/2}\right)$$

$$M \leq M^* \cdot \left(\frac{\ln\left(1/1-\varepsilon\right)}{\ln\left(1/1-\varepsilon/2\right)}\right) + \frac{\ln N}{\ln\left(1/1-\varepsilon\right)}$$

Level III: Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then WM makes $\leq 2.4(M^* + \log_2 N)$ mistakes.

- **Thm**: Any **deterministic** strategy can be forced to make at least $2M^*$ mistakes

Level IV: Randomized Weighted Majority

Give each expert a weight $w_{t,i} \leftarrow 1$, $W_t \leftarrow \sum_i w_{t,i}$

For $t = 1, \ldots, T$:

  Choose i with probability $w_{t,i}/W_t$ (Splitting a dollar across the experts)

  For $i = 1, \ldots, N$:

    If (expert i made a mistake): $w_{t+1,i} \leftarrow (1-\varepsilon) \cdot w_{t,i}$

    Else: $w_{t+1,i} \leftarrow w_{t,i}$, $W_{t+1} \leftarrow \sum_i w_{t+1,i}$

Level IV: Randomized Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then RWM makes $\leq (1+\varepsilon) \cdot M^* + \frac{\log_2 N}{\varepsilon}$ mistakes

- Set $\varepsilon = \sqrt{\frac{\log_2 N}{T}}$

$$M \leq M^* + M^* \sqrt{\frac{\log_2 N}{T}} + \sqrt{T \log_2 N}$$

$$\leq M^* + 2\sqrt{T \log_2 N}$$

$$\frac{M^*}{T} + 2\sqrt{\frac{\log_2 N}{T}}$$

Level IV: Randomized Weighted Majority

- **Thm:** If some expert makes $\leq M$ mistakes then RWM makes $\leq (1+\varepsilon)\cdot M + \frac{\log_2 N}{\varepsilon}$ mistakes

Proof:

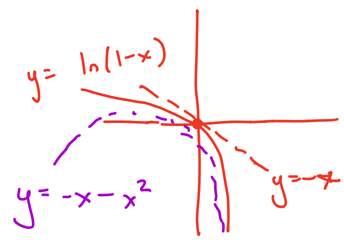
- Measure of progress: W_t
- $(1-\varepsilon)^{M^*} \leq W_T$
- Every day we lose $F_t = \sum_{i : f_{t,i} \neq q_t} \omega_{t,i}$
- $W_{t+1} = W_t (1 - \varepsilon F_t)$ $\quad \therefore \; W_T \leq N \cdot \prod_{t=1}^{T} (1 - \varepsilon F_t)$

$$(1-\varepsilon)^{M^*} \le W_T \le N \cdot \prod_{t=1}^{T} (1-\varepsilon F_t)$$

$$M^* \cdot \ln(1-\varepsilon) \le \ln N + \sum_{t=1}^{T} \ln(1-\varepsilon F_t)$$

$$\le \ln N - \varepsilon \sum_{t=1}^{T} F_t$$

ln(1-x)



what we want to bound

$$\& \sum_{t=1}^{T} F_t \le \frac{-M^* \ln(1-\varepsilon) + \ln N}{\varepsilon}$$

$$= \frac{M^* \ln\left(\frac{1}{1-\varepsilon}\right) + \ln N}{\varepsilon}$$

$$\approx \frac{M^* \ln(1+\varepsilon) + \ln N}{\varepsilon}$$

$$\approx \frac{M^* (\varepsilon + \varepsilon^2) + \ln N}{\varepsilon}$$

$$= (1+\varepsilon) M^* + \frac{\ln N}{\varepsilon}$$

for small x

$\ln(1-x) \approx x$

Level IV: Randomized Weighted Majority

- **Thm:** If some expert makes $\leq M^*$ mistakes then RWM makes $\leq (1+\varepsilon)\cdot M^* + \frac{\log_2 N}{\varepsilon}$ mistakes

Let F_t be the fraction of mistakes we made on day t

Want to bound $\sum_t F_t$ $\qquad F_t \propto \sum_i w_t \, m_{t,i}$

$$(1-\varepsilon)^{M^*} \leq W_T \leq N \cdot \prod_{t=1}^{T} (1-\varepsilon F_t)$$

$$M^* \cdot \ln(1-\varepsilon) \leq \ln N + \sum_{t=1}^{T} \ln(1-\varepsilon F_t)$$

$$\leq \ln N - \sum_{t=1}^{T} \varepsilon F_t$$

$$\sum_{t=1}^{T} F_t \leq \frac{\ln N}{\varepsilon} + \frac{M^*}{\varepsilon \ln(1/1-\varepsilon)}$$

Why I love this algorithm

- **Endless applications:**
  - continuous optimization / linear programming
    - including maximum flow!
  - machine learning
    - training machine learning models
    - combining weak models into strong models
    - online learning: updating models with more data
  - probability theory
  - game theory
    - how to play zero-sum games
  - theory of computation

Why I care so much about this

- We often teach algorithms as a set of *ad hoc* tricks
  - These algorithms are easier to deploy
  - These algorithms are used often
  - These algorithms came first historically
  - These algorithms require less mathematical background
- Algorithms research today is much more systematic
  - More powerful and unified techniques
  - But requires more mathematical sophistication
    - Randomization / Probability / Statistics
    - Continuous Mathematics / Linear Algebra
  - **But beautiful and worth studying!**