Getting Used to Writing and Manipulating LPs

Problem 1 (Generic Linear Programs, 10 points). Suppose you are given a linear program with \(d\) decision variables and \(n = p + q + r\) constraints in the following generic linear program.

\[
\begin{align*}
\max_{x \in \mathbb{R}^d} & \quad \sum_{i=1}^{d} c_i x_i \\
\text{s.t.} & \quad \sum_{i=1}^{d} a_{ij} x_i \leq b_j \quad \forall j = 1, \ldots, p \\
& \quad \sum_{i=1}^{d} a_{ij} x_i = b_j \quad \forall j = p + 1, \ldots, p + q \\
& \quad \sum_{i=1}^{d} a_{ij} x_i \geq b_j \quad \forall j = p + q + 1, \ldots, p + q + r
\end{align*}
\]
(a) Show how to write this LP as an equivalent LP in general form:

\[
\begin{align*}
\max_{x \in \mathbb{R}^d} & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

Keep the size of the general form LP as small as possible.

(b) Write the dual of the generic linear program. What constraints you do you have on the dual variables corresponding to the “\( \leq b_j \)” constraints? The “\( = b_j \)” constraints? The “\( \geq b_j \)” constraints?

**Problem 2** (Linear Regression via Linear Programming, 20 points). In the context of the segmented-least-squares problem we talked about how to take a set of \( n \) points \( \{(x_1, y_1), \ldots, (x_n, y_n)\} \) and represent them by a “line of best fit” of the form \( y = ax + b \). The most common way to define a line of best fit is the one that minimizes an error criterion called the \( L_2 \) error (or squared error),

\[
\varepsilon_2(a, b) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.
\]

However, there are several other natural error criteria leading to different properties, and the corresponding lines of best fit can be found in polynomial time via linear programming.

(a) **(Warm-up, you do not need to hand this in)** For a given \( y \in \mathbb{R} \), write the optimization problem \( \min_{x \in \mathbb{R}} |y - x| \) as a linear program.

(b) **(Warm-up, you do not need to hand this in)** For a given pair \( y_1, y_2 \in \mathbb{R} \), write the optimization problem \( \min_{x \in \mathbb{R}} \max(|x - y_1|, |x - y_2|) \) as a linear program.

(c) The \( L_1 \) error (or total deviation) is defined as

\[
\varepsilon_1(a, b) = \sum_{i=1}^{n} |y_i - ax_i - b|.
\]

Describe a polynomial size linear program whose solution \( (a, b) \) is the line that minimizes \( L_1 \) error.

(d) The \( L_\infty \) error (or maximum deviation) is defined as

\[
\varepsilon_{\infty}(a, b) = \max_{i=1,\ldots,n} |y_i - ax_i - b|.
\]

Describe a polynomial size linear program whose solution \( (a, b) \) is the line that minimizes \( L_\infty \) error.

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Recall, when \( u, v \) are vectors, \( u \leq v \) (resp. \( u \geq v \)) indicates that \( u_i \leq v_i \) (resp. \( u_i \geq v_i \)) for every coordinate \( i \).
Packing and Covering LPs

In the next few problems we will explore two special cases of linear programs called packing and covering linear programs. Packing and covering LPs are nice because their solutions have a certain economic interpretation.

Intuitively, in a packing LP, the decision variables represent items you can choose, the objective function represents a reward for choosing each item, and the constraints say that you cannot choose too many of the items (defined by linear constraints). Your goal is to choose the set of items that maximizes the reward subject to the constraints. For example, our favorite LP from class is a packing LP where the items you can choose are barrels of beer and barrels of ale to produce, the constraints say you cannot choose any set of beer and ale barrels to produce that requires more than 480 corn, 160 hops, or 1190 malt, and the objective says that you receive a reward of $13 for a barrel of ale and $23 for a barrel of beer.

Thus, a generic packing LP can be written as

$$\max_{x \in \mathbb{R}^d} c^T x \quad \text{s.t.} \quad Ax \leq b, \quad x \geq 0$$

where every entry of $c, A, b$ is non-negative. (Note that the transformation in Problem 1 to general form LPs will sometimes produce negative entries, so not every LP is a packing LP.)

Conversely, in a covering LP, the decision variables represent items you can choose, the objective function represents a penalty for choosing each item, and the constraints say that you must choose at least a certain amount of the items (defined by linear constraints). Your goal is to choose the set of items that minimizes the penalty subject to the constraints. For example, the dual LP of our favorite linear program was a covering LP where the variables represented units of corn/hops/malt, the objective said you pay 480 for a unit of corn, 160 for a unit of hops, and 1190 for a unit of malt, and the constraints said that you must buy enough corn/hops/malt to make 13 barrels of ale and 23 barrels of beer.

Thus, a generic covering LP can be written as

$$\min_{y \in \mathbb{R}^n} b^T y \quad \text{s.t.} \quad A^T y \geq c, \quad y \geq 0$$

where every entry of $c, A, b$ is non-negative.

Something magical happened! The dual of a packing LP is a covering LP, and the dual of a covering LP is a packing LP!

**Problem 3** (From Feasibility to Optimization, 20 points). Suppose you are given a packing LP and you don’t know how to solve it in polynomial time. But, you do have a polynomial time algorithm to solve the following feasibility problem. Given $c, A, b, Z$, output a solution $x \geq 0$ such that $c^T x \geq Z$ and $Ax \leq b$, or output that no such solution exists.

Show that you can efficiently approximately solve any packing LP using an algorithm that solves the feasibility problem as a subroutine. That is, if $x^*$ is the optimal solution to the packing LP, you can find a solution $\hat{x}$ such that $c^T \hat{x} \geq c^T x^* - \epsilon$. Bound the running time of your algorithm in terms of the approximation error $\epsilon$, the number of variables and constraints, $n, d$, and the coefficients $c, A, b$. Try to make your algorithm as efficient as possible.
**Problem 4** (Subcovers via Linear Programming, 30 points). Recall the minimum subcover problem from Homework 1. Given a set of intervals \( I_1, \ldots, I_n \) of the form \([a_i, b_i)\) (note that the intervals do not include the right endpoint \( b_i \)), find the smallest subset of \( I \) that forms a cover of the unit interval \([0, 1]\). As before, we assume all the endpoints \( a_1, b_1, \ldots, a_n, b_n \) are distinct.

(a) Show that for every input \( I \), there is a finite set of points \( T \) such that \( S \subseteq I \) is a subcover if and only if \( S \) covers all the points in \( T \).\(^2\)

(b) Formulate the minimum subcover problem as an LP. Write down an LP with decision variables \( x_1, \ldots, x_n \) where \( x_i = 1 \) means that interval \( i \) is in the subcover and \( x_i = 0 \) means that interval \( i \) is not in the subcover.

(c) Write down the dual LP.

(d) Can you find an interpretation of the dual LP?

(e) Prove that both the LP and its dual have integral optimal solutions.\(^3\)

**Problem 5** (Sensitivity of Linear Programs, 30 points). As we’ve seen, you can use linear programming to help your brewery find the optimal amount of beer and ale to produce given a certain supply. But what if the supply is not known exactly, because of uncertainty in our crop yields, or because we had surplus ale last year that we had to drink and aren’t so good at the countin’ right now? It would be bad to mislead our investors about how much money we are going to make this year, so we would like to be able to provide some guarantee that small changes in the supply will not affect our profits too much.

(a) Consider yet again our favorite LP

\[
\text{max } 13A + 23B \quad \text{s.t. } \begin{align*}
5A + 15B &\leq 480 \\
4A + 4B &\leq 160 \\
35A + 20B &\leq 1190 \\
A, B &\geq 0
\end{align*}
\]

and its dual

\[
\text{min } 480C + 160H + 1190M \quad \text{s.t. } \begin{align*}
5C + 4H + 35M &\geq 13 \\
15C + 4H + 20M &\geq 23 \\
C, H, M &\geq 0
\end{align*}
\]

Recall that the optimal primal solution is \((A = 12, B = 28)\) and has revenue 800, and the optimal dual solution is \((C = 1, H = 2, M = 0)\).

Suppose that instead our supply is \(480 + \Delta_C\) units of corn, \(160 + \Delta_H\) units of hops, and \(1190 + \Delta_M\) units of malt. Prove that if we write a new primal LP corresponding to these new supply constraints, then the optimal solution to the new primal LP has revenue at most \(800 + \Delta_C + 2\Delta_H + 0\Delta_M\).

\(^2\)Hint: There is a set \( T \) of size at most \(2n\).

\(^3\)Hint: You may want to use the greedy algorithm from Homework 1 to help you prove this statement.
(b) Now consider a general packing LP with $d$ variables and $n$ constraints,

$$\max_{x \in \mathbb{R}^d} c^T x \quad \text{s.t.} \quad Ax \leq b$$

$$x \geq 0$$

and assume that it’s optimal solution is $x^*$ and has revenue $c^T x^*$. Suppose that instead the constraints are perturbed to obtain the new packing LP

$$\max_{x \in \mathbb{R}^d} c^T x \quad \text{s.t.} \quad Ax \leq b + \Delta$$

$$x \geq 0$$

for some vector $\Delta \in \mathbb{R}^n$. Prove that if $x^\dagger$ is the optimal solution to the new LP, then $c^T x^\dagger \leq \Delta^T y^*$ where $y^*$ is the optimal solution to the dual of the original LP. Thus, if we have already solved the original LP and its dual, we can get a bound on the solution to the new LP (for any $\Delta$) without having to do the work of resolving the LP for each choice of $\Delta$. 