Problem 1 (Asymptotic Growth). Take the following list of functions and arrange them in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list, then it should be the case that $f(n)$ is $O(g(n))$. (You do not need to provide proofs.)

$$
\begin{align*}
  f_1(n) &= n^{2.5} \\
  f_2(n) &= 2^n \\
  f_3(n) &= \sqrt{2}n \\
  f_4(n) &= n + 10 \\
  f_5(n) &= 10^n \\
  f_6(n) &= 100^n \\
  f_7(n) &= n^2 \log n \\
  f_8(n) &= 2^{\sqrt{\log n}} \\
  f_9(n) &= n^{4/3} \\
  f_{10}(n) &= n \log^3 n \\
  f_{11}(n) &= n^{\log n} \\
  f_{12}(n) &= 2^2n
\end{align*}
$$

Problem 2 (Basic Proof Techniques). Use common proof techniques to prove the following statements.

1. A binary tree is a rooted tree in which each node has at most two children. Prove by induction that in any binary tree the number of nodes with two children is exactly one less than the number of leaves.\(^1\)

\(^1\)Hint: Try doing induction on the depth of the tree.
2. Let $G$ be a simple graph on $2n$ nodes with no self-loops (i.e. there are no edges connecting a node to itself and there is at most one edge connecting any pair of nodes). Prove by contradiction that if every node in $G$ has degree at least $n$, then $G$ is a connected graph.

**Problem 3 (Spot the Bug).** One of your classmates tries to convince you that you should let her cheat on her homework because no matter what all students will get the same grade (so you better make sure everyone does well and gets an A). When you look at her skeptically, she provides the following “proof” by induction:

“Proceed by induction on the number of students $n$. (Base case), if $n = 1$, then there is only one student so all students get the same grade. (Inductive step) Assume that any set of $n-1$ students will all get the same grade, and now we will prove that any set of $n$ students will all get the same grade. Let $S$ be a set of $n$ students. Remove one student from $S$ to obtain a set $S_1$ of $n-1$ students. By the inductive hypothesis, all students in $S_1$ get the same grade. Now replace the student you removed and remove a different student to obtain a different set $S_2$ of $n-1$ students. By the same argument, all students in $S_2$ get the same grade. Therefore, all students in $S$ get the same grade and the proof is complete.”

Point out the error in your classmate’s proof.

**Problem 4 (Review of Divide-and-Conquer).** You have been forced to take care of your nephew. Since kids are strange, he demands that you play the following game before he will go to bed. He has a number between 1 and $n$ in his head that you have to guess. If your current guess is closer than the previous guess, he will say “getting closer,” and if it’s farther away he will say “getting farther.” When you get the right answer he will go to bed and you get to watch re-runs of Dr. Who.

Design an algorithm to guess the number quickly. Describe your algorithm in English and in pseudocode (you can use the pseudocode command `guess(x)` to get an answer “closer,” “farther,” or “correct”). Analyze its correctness and running time.

Your algorithm should always find the right answer in $\log_2 n + O(1)$ guesses (it’s not enough to guarantee at most $O(\log n)$ guesses).²

²Hint: it will help to start by thinking about binary search, but you will need a different pattern of guesses to get the smallest number of guesses.