Non-regularity Examples

1. Prove that the language

$$L_1 = \{0^p \mid p \text{ is a prime number}\}$$

is non-regular.

Solution: For the sake of contradiction, assume that L_1 is regular. The Pumping Lemma must then apply; let k be the pumping length. Let n be any prime number at least as large as k (such an n is guaranteed to exist since there are an infinitie number of primes), and consider the string $w = 0^n \in L_1$. Since $|w| \geq k$, it must be possible to split w into three pieces xyz satisfying the conditions of the Pumping Lemma. Now consider the string xy^iz. The string xy^iz has length $n + (i-1)|y|$. Letting $i = n+1$, we have

$$\begin{aligned} n + (i-1)\,|y| &= n + ((n+1)-1)\,|y| \\ &= n + n\,|y| \\ &= n \cdot (1 + |y|) \end{aligned}$$

which is a composite number since $|y| > 0$, and hence xy^iz is not an element of the language. Thus, the Pumping Lemma is violated, and the language in question cannot be regular.

2. Exercise 1.23, part (c). Prove that the language

$$L_2 = \{0^m1^n \mid m \neq n\}$$

is non-regular.

Solution: For the sake of contradiction, assume that L_2 is regular. Then the complement of L_2, $\overline{L_2}$, must also be regular, since the regular languages are closed under complement. $\overline{L_2}$ consists of two types of strings: those strings which are of the form 0^m1^n where $m = n$ and those strings which are not of the form 0^m1^n.

Now consider the language $\overline{L_2} \cap 0^*1^*$. If $\overline{L_2}$ is regular, then $\overline{L_2} \cap 0^*1^*$ must also be regular, since 0^*1^* is regular and the regular languages are closed under interesection. But

$$\overline{L_2} \cap 0^*1^* = \{0^m1^n \mid m = n\} = \{0^n1^n \mid n \geq 0\}$$

which is a known non-regular language (see Example 1.38 on pg. 80 in the Sipser text), a contradiction. Thus, the language L_2 in question cannot be regular.