Non-regularity Examples

1. Prove that the language
\[ L_1 = \{0^p \mid p \text{ is a prime number}\} \]
is non-regular.

Solution: For the sake of contradiction, assume that \(L_1\) is regular. The Pumping Lemma must then apply: let \(k\) be the pumping length. Let \(n\) be any prime number at least as large as \(k\) (such an \(n\) is guaranteed to exist since there are an infinite number of primes), and consider the string \(w = 0^n \in L_1\). Since \(|w| \geq k\), it must be possible to split \(w\) into three pieces \(xyz\) satisfying the conditions of the Pumping Lemma. Now consider the string \(xy^iz\). The string \(xy^iz\) has length \(n + (i - 1)|y|\). Letting \(i = n + 1\), we have
\[
\begin{align*}
n + (i - 1)|y| &= n + ((n + 1) - 1)|y| \\
&= n + n|y| \\
&= n \cdot (1 + |y|)
\end{align*}
\]
which is a composite number since \(|y| > 0\), and hence \(xy^iz\) is not an element of the language. Thus, the Pumping Lemma is violated, and the language in question cannot be regular.

2. Exercise 1.23, part (c). Prove that the language
\[ L_2 = \{0^m1^n \mid m \neq n\} \]
is non-regular.

Solution: For the sake of contradiction, assume that \(L_2\) is regular. Then the complement of \(L_2\), \(\overline{L_2}\), must also be regular, since the regular languages are closed under complement. \(\overline{L_2}\) consists of two types of strings: those strings which are of the form \(0^m1^n\) where \(m = n\) and those strings which are not of the form \(0^m1^n\).

Now consider the language \(\overline{L_2} \cap 0^*1^*\). If \(\overline{L_2}\) is regular, then \(\overline{L_2} \cap 0^*1^*\) must also be regular, since \(0^*1^*\) is regular and the regular languages are closed under intersection. But
\[
\overline{L_2} \cap 0^*1^* = \{0^m1^n \mid m = n\} = \{0^n1^n \mid n \geq 0\}
\]
which is a known non-regular language (see Example 1.38 on pg. 80 in the Sipser text), a contradiction. Thus, the language \(L_2\) in question cannot be regular.