CSU390 Theory of Computation	Extra Credit
Fall 2004	December 7, 2004

Extra Credit Assignment

General Instructions:

1. This is an extra credit assignment; it is completely optional. It will be graded in the same manner as any other homework assignment, except that you are only required to attempt zero problem; i.e., all problems will be graded for extra credit. This assignment will be averaged in with your other homework assignments, as usual.
2. You may choose to submit whole problems or only problem parts. Each problem part will be graded separately.
3. The first three problems were taken from final exams I have given to students in the past. (Collectively, they roughly constitute a single final exam.) As such, these first three problems are good "exam review." The final two problems are more difficult; they were extra credit problems I have given out in the past.
4. If you choose to attempt any or all of this assignment, you must submit your work no later than the time of the final exam for this course.

Problems:	5 problems constituting 9 total problem parts
Required:	zero problem parts
Points:	as stated for each problem or problem part

Problem 1 [30 pts, (10,10,10)]**:** Let L be any language. We define $\textsc{FirstHalf}(L)$ as follows.

$$\textsc{FirstHalf}(L) = \{x \mid \text{there exists } y,\ |y| = |x|, \text{ such that } xy \in L\}\,.$$

In other words, a string x is in $\textsc{FirstHalf}(L)$ if x is the *first half* of some string in L. For example, if

$$L = \{1, 00, 101, 1100, 101001\}$$

then

$$\textsc{FirstHalf}(L) = \{0, 11, 101\}\,.$$

i. Prove that the decidable languages are closed under $\textsc{FirstHalf}$; i.e., show that if L is a decidable language, then $\textsc{FirstHalf}(L)$ is a decidable language.

ii. Prove that NP is closed under $\textsc{FirstHalf}$; i.e., show that if L is in NP, then $\textsc{FirstHalf}(L)$ is in NP.

iii. Your proofs for parts (i) and (ii) almost certainly involve constructing a Turing Machine for FirstHalf(L) given a Turing Machine for L. Can these same constructions be used to show that P is closed under FirstHalf? How or why not?

Note: I am not asking you to prove whether P is closed under FirstHalf or not; rather, I am asking whether *your particular constructions* used in parts (i) or (ii) could also be used to prove that P is closed under FirstHalf. If your proofs for part (i) and/or (ii) *do not* involve TM constructions, then discuss whether your proofs could be adapted to show that P is closed under FirstHalf.

Problem 2 [30 pts, (15,15)]**:** Prove that the following languages are undecidable.

i. $\text{FINITE}_{\mathsf{TM}} = \{\langle M \rangle \mid |L(M)| \text{ is finite}\}$. In other words, (the encoding of) a Turing Machine M is in $\text{FINITE}_{\mathsf{TM}}$ if the number of strings accepted by M is finite.

ii. $101_{\mathsf{TM}} = \{\langle M \rangle \mid L(M) = \{w \mid w \text{ begins with the prefix } 101\}\}$.

Problem 3 [40 pts, (20,20)]**:** Prove that the following problems are NP-complete.

i. Bin Packing. Professor Church was recently fired from his position in the Department of Mathematics at Northeastern University, and he now works part-time at a wharehouse in Boston. The wharehouse has thousands of objects of various sizes[1] which (the former) Prof. Church would like to store in bins of a fixed size. Prof. Church would like to use as few bins as possible since they are expensive, but he is having a difficult time efficiently discerning just how many bins to purchase...

Instance: A finite set of objects X, a size function $s : X \mapsto N$, integers b (bin size) and k (number of bins).

Question: Does there exist a *partition* of X into k disjoint sets $X_1, X_2, \ldots, X_k$ such that for each set X_i

$$\sum_{x \in X_i} s(x) \leq b?$$

Hint: Reduce from Partition. See the appendix of this exam for a description of Partition. Partition was also discussed in class.

ii. Exam Scheduling. Professor Turing was recently fired from his position in the College of Computer and Information Science at Northeastern University, and he now works part-time in the Registrar's Office. The Registrar's Office wants (the former) Prof. Turing to develop an algorithm for scheduling final exams. The Registrar's Office has a list of all courses which have scheduled final exams as well as lists of all students enrolled in each course. Prof. Turing wants to schedule all the exams into as few time periods as possible, with the obvious restriction that no student can be scheduled to take two different exams in the same time period. Prof. Turing is having a difficult time devising an efficient algorithm for this problem...

[1] "Size" in this context could be weight, volume, etc.

Instance: A finite set of courses C, an enrollment list S_c for each $c \in C$, an integer k (number of exam periods).

Question: Does there exist an assignment of the courses to one of k exam periods such that no two courses are assigned to the same exam period if they share a student in their enrollment lists?

Hint: Reduce from 3COLOR. See the appendix of this exam for a description of 3COLOR. 3COLOR is also described in Exercise 7.34 of the Sipser text.

Note: The final two problems are much more difficult; these are not typical in-class "exam" problems.

Problem 4 [30 pts]**:** Prove that P is closed under FIRSTHALF if and only if P = NP.

Hint: The "if" part is easy. (Why?) For the "only if" part, show that there exists a language L in P where FIRSTHALF(L) is NP-complete. (Why does this help?) One can construct such an L from *any* NP-complete language; consider the elements of the NP-complete language and their certificates...

Problem 5 [20 pts]**:** Show that if L_1 and L_2 are both recognizable and $L_1 \cap L_2$ and $L_1 \cup L_2$ are both decidable, then L_1 and L_2 must both be decidable.

Hint: For any string x, what must be true if $x \in L_1 \cap L_2$? What must be true if $x \notin L_1 \cup L_2$? Draw a Venn diagram for L_1 and L_2. The ideas given in the proof of Theorem 4.16 of the Sipser text will help. The proof of this theorem was discussed in class, and the theorem itself is repeated below.

Theorem 1 *A language L is decidable if and only if both L and $\overline{L}$ are recognizable.*

Appendix

PARTITION: Given a collection of integers, is it possible to separate these numbers into two groups so that the groups have the same sum?

Instance: A finite set of integers $Z = \{z_1, z_2, \ldots, z_n\}$.

Question: Can Z be partitioned[2] into two subsets Z_1 and Z_2 such that $\sum_{z \in Z_1} z = \sum_{z \in Z_2} z$?

3COLOR: Given a graph, is it possible to color each of the vertices in this graph with one of three colors so that no edge connects two vertices of the same color?

Instance: Graph $G = (V, E)$.

Question: Does there exist an assignment of the vertices to one of three colors so that no edge connects two vertices of the same color?

[2] In a legal partition, each element of Z must be in either Z_1 or Z_2 (but not both).