Marginal and conditional distributions of multivariate normal distribution

Assume an n-dimensional random vector

\[ \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \]

has a normal distribution \( \mathcal{N}(\mathbf{x}, \mu, \Sigma) \) with

\[ \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \]

where \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are two subvectors of respective dimensions \( p \) and \( q \) with \( p + q = n \). Note that \( \Sigma = \Sigma^T \), and \( \Sigma_{21} = \Sigma_{21}^T \).

**Theorem 4:**

**Part a** The marginal distributions of \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \) are also normal with mean vector \( \mu_i \) and covariance matrix \( \Sigma_{ii} \) (\( i = 1, 2 \)), respectively.

**Part b** The conditional distribution of \( \mathbf{x}_i \) given \( \mathbf{x}_j \) is also normal with mean vector

\[ \mu_{i|j} = \mu_i + \Sigma_{ij} \Sigma_{jj}^{-1} (\mathbf{x}_j - \mu_j) \]

and covariance matrix

\[ \Sigma_{i|j} = \Sigma_{jj} - \Sigma_{ij} \Sigma_{ii}^{-1} \Sigma_{ij} \]

**Proof:** The joint density of \( \mathbf{x} \) is:

\[ f(\mathbf{x}) = f(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{x p\left[ -\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right]} = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{x p\left[ -\frac{1}{2} Q(\mathbf{x}_1, \mathbf{x}_2) \right]} \]

where \( Q \) is defined as

\[ Q(\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \]
Here we have assumed

\[
\Sigma^{-1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1}
\]

According to theorem 2, we have

\[
\Sigma^{11} = (\Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^T)^{-1} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{13} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} \Sigma_{12}^T \Sigma_{11}^{-1}
\]

\[
\Sigma^{22} = (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1} \Sigma_{12} (\Sigma_{12}^2 - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}^T)^{-1} \Sigma_{12} \Sigma_{22}^{-1}
\]

\[
\Sigma^{12} = -\Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22}^2 - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} = (\Sigma^{21})^T
\]

Substituting the second expression for \( \Sigma^{11} \), first expression for \( \Sigma^{22} \), and \( \Sigma^{12} \) into \( Q(x_1, x_2) \) to get:

\[
Q(x_1, x_2) = (x_1 - \mu_1)^T [\Sigma_{11}^{-1} + \Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} \Sigma_{12}^T \Sigma_{11}^{-1}] (x_1 - \mu_1)
\]

\[
-2(x_1 - \mu_1)^T [\Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1}] (x_2 - \mu_2)
\]

\[
+ (x_2 - \mu_2)^T [(\Sigma_{22}^2 - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1}] (x_2 - \mu_2)
\]

\[
= (x_1 - \mu_1)^T \Sigma_{11}^{-1} (x_1 - \mu_1)
\]

\[
+ (x_1 - \mu_1)^T \Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1)
\]

\[
-2(x_1 - \mu_1)^T [\Sigma_{11}^{-1} \Sigma_{12} (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1}] (x_2 - \mu_2)
\]

\[
+ (x_2 - \mu_2)^T [(\Sigma_{22}^2 - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1}] (x_2 - \mu_2)
\]

\[
= (x_1 - \mu_1)^T \Sigma_{11}^{-1} (x_1 - \mu_1)
\]

\[
+ [(x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1)]^T (\Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12})^{-1} [(x_2 - \mu_2) - \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1)]
\]
The last equal sign is due to the following equations for any vectors \( u \) and \( v \) and a symmetric matrix \( A = A^T \):
\[
u^T A u - 2u^T A v + v^T A v = u^T A u - u^T A v + v^T A v
= u^T A(u - v) - (u - v)^T A v = u^T A(u - v) - v^T A(u - v)
= (u - v)^T A(u - v) = (v - u)^T A(v - u)
\]

We define
\[
b \triangleq \mu_2 + \Sigma_{12}^{T} \Sigma_{11}^{-1}(x_1 - \mu_1)
\]

\[
A \triangleq \Sigma_{22} - \Sigma_{12}^{T} \Sigma_{11}^{-1} \Sigma_{12}
\]

and
\[
\begin{align*}
Q_1(x_1) & \triangleq (x_1 - \mu_1)^T \Sigma_{11}^{-1}(x_1 - \mu_1) \\
Q_2(x_1, x_2) & \triangleq [(x_2 - \mu_2) - \Sigma_{12}^{T} \Sigma_{11}^{-1}(x_1 - \mu_1)]^T (\Sigma_{22} - \Sigma_{12}^{T} \Sigma_{11}^{-1} \Sigma_{12})^{-1} [(x_2 - \mu_2) - \Sigma_{12}^{T} \Sigma_{11}^{-1}(x_1 - \mu_1)] \\
& = (x_2 - b)^T A^{-1}(x_2 - b)
\end{align*}
\]

and get
\[
Q(x_1, x_2) = Q_1(x_1) + Q_2(x_1, x_2)
\]

Now the joint distribution can be written as:
\[
f(x) = f(x_1, x_2) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{x^T \Sigma^{-1} x} \exp\left[-\frac{1}{2} Q(x_1, x_2) \right]
\]
\[
= \frac{1}{(2\pi)^{n/2}|\Sigma_{11}|^{1/2}|\Sigma_{22} - \Sigma_{12}^{T} \Sigma_{11}^{-1} \Sigma_{12}|^{1/2}} e^{x^T \Sigma^{-1} x} \exp\left[-\frac{1}{2} Q(x_1, x_2) \right]
\]
\[
= \frac{1}{(2\pi)^{p/2}|\Sigma_{11}|^{1/2}} e^{x_1^T \Sigma_{11}^{-1} x_1} \frac{1}{(2\pi)^{q/2}|A|^{1/2}} e^{x_2^T A^{-1} x_2} \exp\left[-\frac{1}{2} (x_2 - b)^T A^{-1}(x_2 - b) \right]
\]
\[
= N(x_1, \mu_1, \Sigma_{11}) N(x_2, b, A)
\]

The third equal sign is due to theorem 3:
\[
|\Sigma| = |\Sigma_{11}| |\Sigma_{22} - \Sigma_{12}^{T} \Sigma_{11}^{-1} \Sigma_{12}|
\]
The marginal distribution of $\mathbf{x}_1$ is

$$f_1(x_1) = \int f(x_1, x_2) \, dx_2 = \frac{1}{(2\pi)^{p/2} |\Sigma_{12}|^{1/2}} \exp\left(-\frac{1}{2}(x_1 - \mu_1)^T \Sigma_{11}^{-1}(x_1 - \mu_1)\right)$$

and the conditional distribution of $\mathbf{x}_2$ given $\mathbf{x}_1$ is

$$f_{2|1}(x_2|x_1) = \frac{f(x_1, x_2)}{f(x_1)} = \frac{1}{(2\pi)^{q/2} |A|^{1/2}} \exp\left(-\frac{1}{2}(x_2 - b)^T A^{-1} (x_2 - b)\right)$$

with

$$b = \mu_2 + \Sigma_{12}^T \Sigma_{11}^{-1} (x_1 - \mu_1)$$

$$A = \Sigma_{22} - \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12}$$