Linear programming duality

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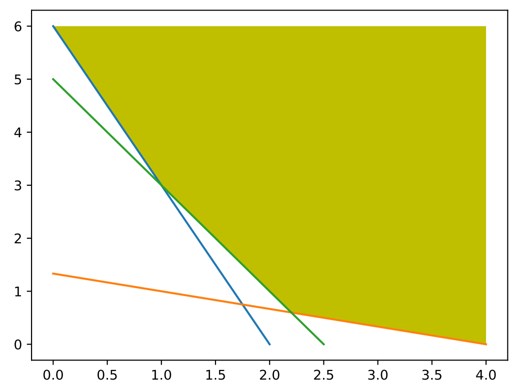
1 Duality

An important question in linear programming is to be able to certify the optimality of the solution. Consider a small example:

$$\min x + y \text{ subject to}$$
$$3x + y \geq 6$$
$$x + 3y \geq 4$$
$$2x + y \geq 5$$
$$x, y \geq 0$$



If the software gives us the solution $x = 11/5, y = 3/5$, how good is this solution? This question is intimately connected to the question of giving an lower bound on the solution.

One possible lower bound comes from adding up 1/4 times the first inequality plus 1/4 times the second inequality:

$$x + y = \frac{1}{4}(3x + y) + \frac{1}{4}(x + 3y) \geq \frac{1}{4} \cdot 6 + \frac{1}{4} \cdot 4 = \frac{5}{2}$$

Question: can you find a better lower bound?

This question brings us to the concept of duality. For each linear program, there is an associated dual linear program:

Primal	**Dual**
$\min c^T x$	$\max b^T y$
$Ax \geq b$	$A^T y \leq c$
$x \geq 0$	$y \geq 0$

Notice that for each constraint in the original problem, there is a corresponding variable in the dual problem. Similarly, for each variable in the original problem there is a corresponding constraint in the dual problem.

Question. Write the dual linear programs for the assignment problem and the shortest path problem.

Notice that every solution for the dual linear program gives a lower bound on the value of the primal problem.

Theorem 1.1 (Weak duality)**.** *Consider an arbitrary feasible solution y for the dual LP and an arbitrary feasible solution x for the primal LP. we have*

$$x^T c \geq y^T b$$

Proof. Because x is a feasible solution for the primal LP:

$$Ax \geq b$$

Because $y \geq 0$, we can take dot product of both sides with y and obtain $y^T Ax \geq y^T b$.

Similarly, because y is a feasible solution for the dual LP:

$$A^T y \leq c$$

Because $x \geq 0$, we can take dot product of both sides with x and obtain $x^T A^T y \leq x^T c$

Combining the two inequalities, we obtain $x^T c \geq y^T b$ □

Theorem 1.2 (Strong duality)**.** *If the primal and dual problems are feasible then their optimal values are equal.*

To prove the theorem, we will make use of a useful theorem:

Lemma 1.3 (Separating hyperplane theorem)**.** *Let P be a closed convex set and x be a point not in P. There exists a vector w such that $w^T x > \max_{z \in P} w^T z$.*

The theorem is intuitive but proving it requires some formal math so we will skip it. We now proceed to prove the duality theorem.

Lemma 1.4. *Let x^* be the optimal solution for the primal LP. Let S be the set of constraints j that are tight i.e. $(Ax^*)_j = b_j$. There exist $\{\lambda_j \geq 0\}_{j \in S}$ such that $c_i = \sum_{j \in S} \lambda_j A_{ji}$ for all i.*

Proof. Suppose for contradiction that no such $\{\lambda_j\}$ exist. Let A_j denote row j of the constraint matrix A. Let

$$P = \left\{ v \middle| v = \sum_{j \in S} \lambda_j A_j \text{ for some } \{\lambda_j \geq 0\}_{j \in S} \right\}$$

i.e. P is the set of all linear combinations with nonnegative coefficients of the rows of A in S.

Observe that P is closed and convex (why?) and by our assumption, $c \notin S$ so there exists some w such that $w^T c > \max_{v \in P} w^T v$. Note that this means $w^T c > 0$ and $w^T A_j \leq 0$ $\forall j \in S$ (why?).

Consider the vector $x - \varepsilon w$ for a tiny positive constant ε. We will show that this is a feasible solution with better objective value than x, which is a contradiction:

- For constraint $j \notin S$, because $A_j^T x > b_j$ and ε is sufficiently small, $A_j^T (x - \varepsilon w) > b_j$. For constraint $j \in S$, we have $A_j^T (x - \varepsilon w) = b_j - \varepsilon A_j^T w b_j \geq b_j$ because $A_j^T w \leq 0$.

- The objective value decreases since $c^T (x - \varepsilon w) = c^T x - \varepsilon c^T w < c^T x$.

□

Thus, the objective coefficients is a conic combination of the coefficients in the constraints in S. Consider a set of values for $\lambda_j \geq 0$ so that $c = \sum_j \lambda_j A_j$ and set $\lambda_j = 0 \ \forall j \notin S$.

Observe that

- $\lambda \geq 0$
- $A^T \lambda = \sum_j \lambda_j A_j = c$
- $b^T \lambda = \sum_{j \in S} b_j \lambda_j = \sum_{j \in S} (x^T A_j) \lambda_j = x^T c$

Thus, λ is a solution to the dual problem with dual objective value exactly equal to the optimal primal objective value.

2 Special cases of the duality theorem

There are many interesting special cases of the duality theorem for linear programming. We will mention an example, which many of you might have seen in an undergraduate course.

Consider the maximum flow problem. We are given a directed graph $G = (V, E)$ with source s and sink t. Each edge e has a capacity c_e. The flow on each edge must be at most its capacity and at any vertex other than s, t, the flow must be conserved: the total incoming flow must be equal to the total outgoing flow. We would like to maximize the total flow we can send from s to t.

Let's formulate this problem as a linear program. Let P be the set of directed simple paths from s to t. Let x_p be the variable measuring the amount of flow we are sending on the path p. We have

$$\max \sum_{p \in P} x_p :$$

$$\sum_{p: e \in p} x_p \leq c_e \ \forall e \in E$$

$$x_p \geq 0 \ \forall p \in P$$

Let's write the dual of this linear program.

$$\min \sum_{e \in E} c_e y_e :$$

$$\sum_{e \in p} y_e \geq 1 \ \forall p \in P$$

$$y_e \geq 0 \ \forall e \in E$$

Notice that this dual represents a fractional version of the minimum cut problem: each edge is picked up to a fraction y_e with the constraint that on any path from s to t, the total fraction of edges being picked is at least 1. The usual minimum cut problem restricts the variables y_e to be either 0 or 1. It turns out that this LP also has an optimal integral solution so its value is equal to the value of a cut in the graph.

Thus, LP duality implies that the maximum flow is equal to the capacity of the minimum cut.