H2.

- The assignment is due at Gradescope on Friday, January 27 at 11:59am. Late assignments will not be accepted. Submit early and often.
- You are permitted to study with friends and discuss the problems; however, *you must write up you own solutions, in your own words*. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please do list all your collaborators in your submission for each problem.
- Finding solutions to homework problems on the web, or by asking students not enrolled in the class is strictly prohibited.
- We require that all homework submissions are prepared in Latex. If you need to draw any diagrams, however, you may draw them with your hand. Please use *a new page to begin each answer.*

problem 1 *Flip-flop*

Consider the recurrence $T(n) = 2T(n/2) + f(n)$ in which

$$f(n) = \begin{cases} n^3 & \text{if } \lceil \log(n) \rceil \text{ is even} \\ n^2 & \text{otherwise} \end{cases}$$

(a) We will attempt to use master theorem to solve this recurrence. What are a and b? Show that $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ for some $\epsilon > 0$.

Solution:

(b) Explain why the third case of the Master's theorem does not apply.

Solution:

(c) Prove that $T(n) = O(n^3)$ by induction.

Solution:

(d) Prove that $T(n) = \Omega(n^3)$ by induction.

Solution:

PROBLEM 2 *Trolley*

The city of Boston commissions you to design a new bus system for Huntington Avenue which has n stops numbered $1, 2, \ldots, n$ on the North-bound route (lets ignore the South-bound route). Commuters may begin their journey at any stop i and end at any other stop $j > i$. There are some obvious options:

1. You can have a bus run from the southern-most point to the northern-most point as a traditional busline might run. The system would be cheap because it only requires n segments for the entire system. However, a person traveling from stop $i = 0$ to stop $j = n$ must travel through all n segments. This system will be slow for that person.

2. You can have a special express bus run from every point to every other destination. No person will every wait through any unnecessary segments no matter where they start and end. However, this system requires $\Theta(n^2)$ segments and will be expensive.

We will design a compromise solution: use a divide-and-conquer technique to design a bus system that uses $\Theta(n \log n)$ route segments and which requires a person to wait through at most 1 extra segment when going from any i to any j (as long as $i \leq j$, i.e., we only consider North-bound routes for simplicity, and all buses run North). In other words, a commuter can travel from any i to any j by using at most 2 segments.

(a) For the base cases $n = 1, 2$, design a system using at most 1 route.

Solution:

(b) For $n > 2$ we will use devide-and-conquer. Assume that we already put in place routes connecting the first $n/2$ stops and routes connecting the last $n/2$ stops so that if i and j both belong to the same half, we can get from i to j in at most 2 segments. Show how to add $O(n)$ additional routes so that if i is in the first half and j is in the second half we can get from i to j in 2 segments.

Solution:

(c) Write the recurrence for the number of routes your solution use and solve it using the master theorem.

Solution:

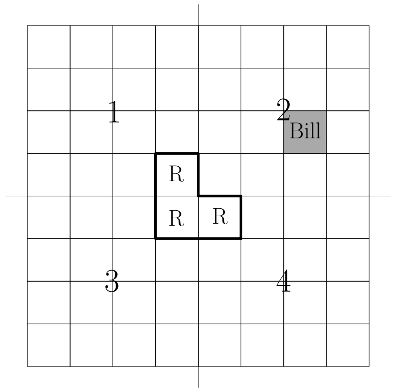

Figure 1: A $2^3 \times 2^3$ floor with one square occupied by Bill's statue. The rows are from 1 to 2^3 and the columns are also from 1 to 2^3.

PROBLEM 3 *Tiling*

We are retiling the floor of West Village H, which has dimension $2^n \times 2^n$, and one square is reserved to be occupied by the statue of a wealthy donor (who we will refer to as "Bill"). The location of the statue can be anywhere on the floor as the donor desired. The rest of the squares are tiled by L-shape pieces, each composed of 3 squares.

(a) Design an algorithm using divide and conquer such that given $n \geq 0$ and the location of the statue at row a and column b, the algorithm can output a list of the locations and orientations of $(4^n - 1)/3$ tiles. Hint: divide into quadrants (marked 1,2,3,4 in the figure) and reserve one center square in each of the 3 quadrants without the statue (marked R in the figure).

Solution:

(b) Write a recurrence for the running time and solve it using the master theorem. Hint: you might need to change to a new variable $m = 2^n$.

Solution: