H1.

- The assignment is due at Gradescope on January 20 at 11:59am. Late assignments will not be accepted. Submit early and often.
- You are permitted to study with friends and discuss the problems; however, *you must write up you own solutions, in your own words*. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please do list all your collaborators in your submission for each problem.
- Finding solutions to homework problems on the web, or by asking students not enrolled in the class is strictly prohibited.
- We require that all homework submissions are prepared in Latex. If you need to draw any diagrams, however, you may draw them with your hand.

PROBLEM 1 *Asymptotic Notation Review*

Rank the following functions by order of growth; that is, find an ordering $f_1, f_2, \ldots, f_{12}$ of the functions satisfying $f_i = \Omega(f_{i+1})\ \forall i \in \{1, \ldots, 11\}$. You do not need to provide justification. Hint: use logarithms to simplify the functions.

$$\begin{array}{cccccc} n^3 & 7^{\log_4 n} & n! & 2^{(\log_2 n)^2} & \log_2(n!) & n^{1/(\log_2 n)} \\ \log_2 \log_2 n & (\log_2 n)^{(\log_2 n)/(\log_2 \log_2 n)} & \sqrt{n} & 2^{\log_3 n} & 3^{\sqrt{\log_2 n}} & 1.01^n \end{array}$$

Solution:

PROBLEM 2 *Faulty induction*

Consider the following theorem and proof.

Theorem 1 *In every set of $n \geq 1$ huskies, all huskies have the same color.*

Proof. In the base case $n = 1$, because the set has only one husky, it has the same color as itself. In the inductive step, assume that the theorem is true for $n = k$. We will prove the theorem for $n = k + 1$. Consider a set of $k + 1$ huskies $a_1, \ldots, a_k, a_{k+1}$. By our assumption, the first k huskies have the same color.

$$\underbrace{a_1, a_2, \ldots, a_k}_{\text{same color}}, a_{k+1}$$

Also by our assumption, the last k huskies also have the same color.

$$a_1, \underbrace{a_2, \ldots, a_k, a_{k+1}}_{\text{same color}}$$

Therefore, by transitivity, all huskies in the set have the same color.

Thus, by induction, the theorem is true for all $n \geq 1$. $\square$

Is there any mistake in this proof or is math broken?

Solution:

PROBLEM 3 *Karatsuba Example*

Carry out the Karatsuba algorithm for $12 \cdot 98$.

Solution:

PROBLEM 4 *Mysterious code*

You encounter the following mysterious piece of code in a cryptographic program.

```
function Compute(a, n)
    if n = 0 then
        return the pair (1, a)
    else if n = 1 then
        return the pair (a, a · a)
    else if n even then
        (u, v) ←Compute(a, ⌊n/2⌋)
        return (u · u, u · v)
    else if n odd then
        (u, v) ←Compute(a, ⌊n/2⌋)
        return (v · u, v · v)
    end if
end function
```

(a) What is the result of Compute$(a, 3)$?

Solution:

(b) What is the result of Compute$(a, 4)$?

Solution:

(c) What does the code do in general? Prove your assertion by induction on n.

Solution:

(d) Write the recurrence for the running time as a function of n and show $T(n) = O(\log n)$ by induction. Assume that each multiplication takes one unit of time and you only need to count the number of multiplications.

Solution: