

Problem Set 5

Due: **Monday 4 August 2014** in class.

1. (20 points) Sketch the following vector fields. Are they conservative? If so, find a potential function.

(a) $\mathbf{F}(x, y) = \mathbf{i} + x\mathbf{j}$

(b) $\mathbf{F}(x, y) = \langle y - x, x - y \rangle$.

(c) $\mathbf{F}(x, y) = \frac{\langle 1-y, x-2 \rangle}{\sqrt{(x-2)^2 + (y-1)^2}}$ Hint: How are the vectors $\langle a, b \rangle$ and $\langle -b, a \rangle$ related?

2. (20 points) Let $\mathbf{F}(x, y) = \langle \cos y, -x \sin y \rangle$. Let C be the line segment connection $(0, 0)$ to $(2, \pi)$.

(a) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly. Feel free to look up any antiderivatives.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by the fundamental theorem of line integrals

3. (20 points) *Area of a polygon*

(a) Use Green's theorem to show that the area inside the closed, positively oriented curve C is given by both $\oint_C x dy$ and $-\oint_C y dx$.

(b) Suppose the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ specify a positively oriented polygon. By computing one of the line integrals from (a), show that the area of the polygon is

$$\sum_{i=1}^n \frac{x_{i+1} + x_i}{2} (y_{i+1} - y_i),$$

where we interpret (x_{n+1}, y_{n+1}) to mean (x_1, y_1) .

4. (20 points) *Impossibility of particle trapping by electrostatic forces.* Away from point charges, the 2d electric potential $\phi(x, y)$ satisfies

$$\partial_{xx}\phi(x, y) + \partial_{yy}\phi(x, y) = 0 \tag{*}$$

If $(0, 0)$ is a local minimum of ϕ , then a particle located at $(0, 0)$ will be trapped in a stable manner. We will prove that this is impossible. Our approach is to show that any function ϕ satisfying (*) obeys the mean value property: $\phi(0, 0)$ equals the average value of ϕ over any circle centered at the origin. Hence, unless ϕ is constant, ϕ takes values both greater than and less than $\phi(0, 0)$ along every circle centered at the origin. Hence $\phi(0, 0)$ can not be a local minimizer.

(a) Let C be the positively oriented unit circle, centered at the origin. The line integral $f(r) = \frac{1}{2\pi} \int_C \phi(rx, ry) ds$ is the average value of ϕ on the circle of radius r centered at the origin. Compute $f'(r)$ and express it in the form $\frac{1}{2\pi} \int_C \mathbf{F}(x, y) \cdot \langle -y, x \rangle ds$.

(b) Because the unit tangent vector to C is $\langle -y, x \rangle$, we can write $f'(r) = \frac{1}{2\pi} \int_C \mathbf{F} \cdot \langle dx, dy \rangle$. Use Green's theorem to show that f is constant. Hence, continuity of ϕ guarantees that the average value of ϕ over any circle centered at the origin is $\phi(0, 0)$.

(c) Could we have used the second derivative test to conclude that ϕ has no local minimizers? Why or why not?