7 July 2014
Calculus 3, Interphase 2014
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Problem Set 2

Due: **14 July 2014** in class.

1. (10 points)

   (a) Let $\mathbf{a}(t)$ and $\mathbf{b}(t)$ give curves in 3d. By writing out the components explicitly, show that
   $$\frac{d}{dt}\Big(\mathbf{a}(t)\cdot\mathbf{b}(t)\Big) = \frac{d\mathbf{a}}{dt}\cdot\mathbf{b} + \mathbf{a}\cdot\frac{d\mathbf{b}}{dt}$$

   (b) Show that if the speed of an object is constant, then its acceleration is always perpendicular to its velocity.
   Hint: If the object has velocity $\mathbf{v}(t)$, study $\frac{d}{dt}|\mathbf{v}(t)|^2$.

2. (20 points) Parameterizations

   (a) Parameterize the square traversed from $(0,0)$ to $(0,1)$ to $(1,1)$ to $(1,0)$ and back to $(0,0)$. Your answer should be expressed as a piecewise linear function $\mathbf{X}(t)$.

   (b) A gear of radius R is centered at the origin and fixed so that it can not move or rotate. Another gear of radius r initially touches the fixed gear at $(-R,0)$ and rotates clockwise around the fixed gear. Sketch and find the trajectory of the point originally touching the fixed gear.

3. (15 points) Let $\mathbf{X}(t) = (e^{-t}\cos t, e^{-t}\sin t)$ for $0 \le t < \infty$.

   (a) Plot this curve.

   (b) Compute the velocity vector and speed as functions of t.

   (c) Compute the length of the curve.

   (d) How many times does the curve circle the origin?

   (e) Does the contrast of (c) and (d) surprise you?

   (f) Extra credit (5 points): Find a curve of the form $(r(t)\cos t, r(t)\sin t)$ that spirals into the origin and has infinite arc length. Justify your answer with a calculation.

4. (20 points) Solutions to partial differential equations

   (a) Let $u(t,x)$ be the pressure of air at time t and 1d position x. Pressure disturbances represent sound and travel according to the wave equation:

   $$\partial_{tt}u(t,x) - c^2\partial_{xx}u(t,x) = 0.$$

   Show that $u(t,x) = \sin(x - ct)$ satisfies the wave equation. Assuming $c > 0$, in which direction is the wave moving? At what speed? At any fixed x, what is the frequency of oscillation? This frequency corresponds to the tone that a fixed observer would hear.

   (b) Let $T(t,x)$ be the temperature at time t and position x along a long metal rod. If the rod is initially heated at the origin, the heat will spread in both directions according to the heat equation:

   $$\partial_t T(t,x) - \partial_{xx}T(t,x) = 0.$$

   Show that $T(t,x) = \frac{1}{\sqrt{4\pi t}}\exp(-\frac{x^2}{4t})$ satisfies the heat equation for any $t > 0$ and $-\infty < x < \infty$. Sketch the function T at various times t. The x^2/t term in the exponent reveals that in time t, heat travels a distance proportional to $\sqrt{t}$. That is to say, to get the heat to travel twice as far, you have to wait four times as long!

5. (9 points) Sketch the level curves and the three-dimensional surfaces given by

   (a) The saddle: $f(x,y) = x^2 - y^2$

   (b) The 2d bell curve: $f(x,y) = e^{-x^2-y^2}$

   (c) The gravitational potential of a 2d planet: $f(x,y) = \log\sqrt{x^2+y^2}$

6. (10 points) Consider the ellipsoid defined by $x^2 + 2y^2 + 3z^2 = 9$.

   (a) Find the tangent plane at the point $(2,-1,1)$ by viewing the surface as a level set of a function of three variables.
   Hint: How are gradients and level sets related?

   (b) Use the plane from (a) to approximate the value of z on the surface at the coordinates $x = 2.01$, $y = -0.99$. Compute the exact value of z and determine how much error is in the approximation.

7. Consider a hill with height given by $z = x^2 - y^2$. Suppose a hiker is initially at $(2,1,3)$ and always walks in the direction of steepest ascent.

   (a) (6 points) Sketch the level curves of z and the path taken by the hiker.

   (b) (10 points) Initially, what is the 3d vector tangent to the hiker's path?